AN ALGEBRAIC GEOMETRIC REALIZATION OF THE CHERN CHARACTER

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CONTENTS

1. Introduction

In this paper we provide a presentation, in the level of classifying spaces, of the Chern character and its relation to Chern classes. Although this is apparently a classical and well-known subject, our constructions have the important feature of preserving the algebraic geometric nature of the objects involved. The first manifestation of this lies in the fact is that all spaces and maps involved are colimits of directed systems in the category $\mathfrak{Var}_{\mathbb{C}}$ of projective algebraic varieties and algebraic maps. We will then explain the relevance of this fact for the study of the morphic cohomology introduced by E. Friedlander and B. Lawson in [FL92] and of holomorhic K-theory performed by Cohen and Lima-Filho in [CLF]. We expect that the constructions made here, as well as in [CLF], can be extended to a broader context, such as [Fri97], once appropriate facts in motivic rational homotopy theory are in place.

The constructions of classifying spaces made, together with their rationalizations, involve three different algebraic geometric moduli spaces: symmetric products of projective spaces $\mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n))$, Grassmannians $\mathrm{Grass}^q(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)$ and Chow varieties $\mathrm{Chow}_d^q\big(\mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)\big)$ of projective spaces. Here we use $\mathbb{C}^{\vee n}$ to denote the dual of $\mathbb{C}^n$. A fundamental link between these spaces are the quotient varieties $\mathrm{Grass}^q(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)/\mathfrak{S}_q$, consisting of Grassmannians modulo the natural action of the symmetric group $\mathfrak{S}_q$. These *symmetrized Grassmannians* also appear in the study of Conformal Field Theory.

The departing point is the observation, made in Section 2, that the natural direct sum map $\mathbb{P}(\mathbb{C}^n)^{\times q} = \mathrm{Grass}^1(\mathbb{C}^{\vee n})^{\times q} \to \mathrm{Grass}^q(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)$ descends to an algebraic map $\mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n)) \to \mathrm{Grass}^q(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)/\mathfrak{S}_q$. These maps fit into a directed system of algebraic maps and varieties and induce a map

$$f : \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)) \to \mathrm{BU}/\mathfrak{S}_\infty$$

between the corresponding colimits. Section 2 is mostly devoted to setting up the directed systems, understanding various "algebraic geometric" filtrations giving the topology of $\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))$ and $\mathrm{BU}/\mathfrak{S}_\infty$. The infinite symmetric product $\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))$ has a classical structure of an abelian topological monoid whose addition is an algebraic map, and we show in Proposition 2.8 that so is $\mathrm{BU}/\mathfrak{S}_\infty$. We conclude Section 2 by proving in Proposition 2.15 that $f : \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)) \to \mathrm{BU}/\mathfrak{S}_\infty$ is a monoid morphism and a rational homotopy equivalence.

Our guiding principle is to see colimits of directed systems of type $\{M_\lambda\}_{\lambda \in \Lambda}$, with M_λ an algebraic variety, as representing functors from the category of algebraic varieties to the category of topological spaces. In particular, whenever they come equipped with algebraic

maps making the colimit $M_\infty = \varinjlim_\lambda M_\lambda$ into a topological monoid, then M_∞ represents a functor $X \mapsto \mathfrak{Mor}(X, M_\infty)$ from the category of algebraic varieties to the category of topological monoids. In particular, all objects and maps in Section 2 must be seen as representing, respectively, functors and natural transformations in this context.

We start Section 3 by introducing a method of constructing the $\mathbb{Q}$-localization of an abelian topological monoid in the category of projective varieties. In particular we present natural directed systems in $\mathfrak{Var}_{\mathbb{C}}$ to produce rationalization maps $r_P : \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)) \to \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q}$ and $r_B : BU \to \{BU/\mathfrak{S}_\infty\}_\mathbb{Q}$, and think of the latter as a model for $BU_\mathbb{Q}$ in view of Proposition 2.8. In these $\mathbb{Q}$-local topological monoids, multiplication by an integer is an invertible algebraic map. This localization scheme then can be used to discuss monoids of the type $\mathfrak{Mor}(X, M_\infty)$ as above, along with their group completions $\mathfrak{Mor}(X, M_\infty)^+$; cf. Appendix A. In particular, we discuss the canonical splittings of $\mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^n))^+$ and their rationalizations.

We then study the functors represented by the constructions of Section 2 and their rationalization. First we consider the case $X = \{pt\}$. We use the fact, shown in Proposition 2.8, that the projection $\rho : \mathrm{BU} \to \mathrm{BU}/\mathfrak{S}_\infty$ is a rational equivalence, to identify $H^*(\mathrm{BU}, \mathbb{Q})$ with $H^*(\mathrm{BU}/\mathfrak{S}_\infty, \mathbb{Q})$ via ρ^*. Then we denote by $\mathbf{i}_{2j}$ the image in rational cohomology of the integral class represented by the composition

$$\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)) \xrightarrow[sp]{\simeq} \prod_{j=1}^\infty K(\mathbb{Z}, 2j) \xrightarrow[\mathrm{pr}_j]{} K(\mathbb{Z}, 2j),$$

where sp is the canonical splitting equivalence presented in [FL92], and pr_j is the projection. The first main result is the following.

Theorem 3.7. *Let $f : SP_\infty(\mathbb{P}(\mathbb{C}^\infty)) \to BU/\mathfrak{S}_\infty$ be the homomorphism of Proposition 2.15, and identify $H^*(BU/\mathfrak{S}_\infty; \mathbb{Q})$ with $H^*(BU; \mathbb{Q})$ via the projection $\rho : BU \to BU/\mathfrak{S}_\infty$. Then, for $j \geq 1$ one has*

$$f^*(\, j!\, \mathbf{ch}_j \,) \;=\; \mathbf{i}_{2j},$$

where $\mathbf{ch}_j$ is the $2j$-th component of the universal Chern character $\mathbf{ch} \in H^*(BU; \mathbb{Q})$.

Therefore, it follows from Theorem 3.7, each Chern character can be realized by algebraic maps. Alternatively, these localizations are natural and so they give rise to a homomorphism

$f_{\mathbb{Q}} : \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_{\mathbb{Q}} \to BU_{\mathbb{Q}}$ which makes the square in following diagram commute.

We produce a homotopy equivalence $\mathbf{e} : \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_Q \to \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_Q$ and define $\mathbf{ch} : BU \to \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_Q = \prod_{j=1}^\infty K(\mathbb{Q}, 2j)$ by $\mathbf{ch} = \mathbf{e} \circ f_{\mathbb{Q}}^{-1} \circ \rho$, where $f_{\mathbb{Q}}^{-1}$ is a homotopy inverse for $f_{\mathbb{Q}}$. The equivalence $\mathbf{e}$ is chosen so that the following result holds.

Theorem 3.8. *Let* $\mathbf{ch} : BU \to SP_\infty(\mathbb{P}(\mathbb{C}^\infty))$ *be the composition* $\mathbf{ch} := \mathbf{e} \circ f_{\mathbb{Q}}^{-1} \circ \rho_B$. *Then* $\mathbf{ch}$ *represents the Chern character. In other words,* $\mathbf{ch}^*(\mathbf{i}_{2j}) = \mathbf{ch}_{2j} \in H^{2j}(BU; \mathbb{Q})$.

We then proceed to study the case X arbitrary. Here we must initially understand the functors represented by our constructions. We first observe that the functor $X \to \mathcal{M}\mathrm{or}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+$ represents the morphic cohomology of X. More precisely,

$$\pi_i(\ \mathcal{M}\mathrm{or}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+) \cong \prod_j L^j H^{2j-i}(X),$$

where $L^p H^n(X)$ are the morphic cohomology groups introduced by Friedlander and Lawson in [FL92]; see (36). Then we introduce the spaces $\tilde{\mathcal{K}}_{hol}(X) := \mathcal{M}\mathrm{or}(X, BU)^+$, called the (reduced) *holomorphic K-theory space* of X, and define the (reduced) *holomorphic K-theory groups* of X as

$$\tilde{\mathcal{K}}_{hol}^{-i}(X) := \pi_i(\tilde{\mathcal{K}}_{hol}(X));$$

cf. Definition 3.14. These groups are also studied by Friedlander and Walker in [FW99]. The main result states that the algebraic maps between the classifying spaces under consideration still induce uniquely determined homotopy class of maps between the corresponding represented objects. In particular, we prove the following.

Theorem 3.16. *Let* X *be a projective variety. The natural maps* $\mathbf{ch}^X : \mathcal{M}\mathrm{or}(X, BU)^+ \to \mathcal{M}\mathrm{or}(X, SP_\infty(\mathbb{P}(\mathbb{C}^\infty)))_{\mathbb{Q}}^+$ *induce natural homomorphisms*

$$\mathbf{ch}_X^i : \tilde{\mathcal{K}}_{hol}^{-i}(X) \to \prod_{j \geq 0} L^j H^{2j-i}(X)_{\mathbb{Q}}$$

from the holomorphic K-theory groups of X to its rational morphic cohomology. These homomorphism fit into a commutative diagram

$$
\begin{array}{ccc}
\tilde{\mathcal{K}}_{hol}^{-i}(X) & \longrightarrow & \tilde{\mathcal{K}}_{top}^{-i}(X) \\
\mathbf{ch}_X^i \downarrow & & \downarrow \mathbf{ch}_X^i \\
\prod_{j\geq 0} L^j H^{2j-i}(X)_{\mathbb{Q}} & \longrightarrow & \prod_{j\geq 0} H^{2j-i}(X,\mathbb{Q}),
\end{array}
$$

where the right vertical arrow is the usual Chern character from topological K-theory to ordinary cohomology category, cf. Theorem 3.8, and the horizontal arrows are given by the usual forgetful functors.

In Section 4 we discuss the Chow varieties $\mathcal{C}_{n,d}^q = \mathrm{Chow}_d^q\big(\mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)\big)$ parametrizing effective algebraic cycles in projective spaces, and their relation to the Chern classes in the present context. We start considering the case $X = \{pt\}$, and this is essentially a survey of fundamental results of Lawson [Law89], Lawson and Michelsohn [LM88], Boyer, Lawson, Lima-Filho, Mann and Michelsohn [BLLF⁺93] and Lima-Filho [LF99]. These form a directed system $\{\mathcal{C}_{n,d}^q; t_{n,(d,e)}^q, \epsilon_{d,n}^{(q,k)}, s_{(n,m),d}^q\}$, whose colimit $\mathcal{C}$ has a canonical splitting homotopy equivalence $\mathcal{C} \simeq \prod_{j=1}^{\infty} K(\mathbb{Z}, 2j)$; cf. [Law89]. An important fact is that this equivalence comes from an equivalence between $\mathcal{C}$ and $\mathrm{SP}_{\infty}(\mathbb{P}(\mathbb{C}^{\infty}))$ given by Lawson's complex suspension theorem [Law89], hence one can use $\mathcal{C}$ as a classifying space for morphic cohomology in the category of varieties; cf. [FL92]. There is a natural map $c : BU \to \mathcal{C}$ which represents the total Chern class [LM88], and $\mathcal{C}$ has an infinite loop space structure so that c is an infinite loop space map.

In the case $X \neq \{pt\}$, a subtle issue related to the group-completion functor requires the smoothness of X in order to define a Chern class map

$$
\mathbf{c}_X \;:\; \mathfrak{Mor}(X, BU)^+ \to\; \mathfrak{Mor}(X, \mathcal{C}_{\infty,*}^{\infty})_1^+.
$$

This map induces higher Chern class maps

$$
\mathbf{c}_X^i \;:\; \tilde{\mathcal{K}}_{hol}^{-i}(X) \to \prod_{p\geq 1} L^p H^{2p-i}(X);
$$

cf. Definition 4.12.

In Section 5 we discuss relations between the Chern classes and the Chern character. The starting point is the natural averaging map $\mathbf{av}^q : \mathcal{C}_{n,d}^q \to \mathcal{C}_{n,dq!}^q$, coming from the evident action of the symmetric group $\mathfrak{S}_q$. These maps assembles to give a natural rationalization

map $\mathbf{av}^\infty : \mathcal{C} \to \mathcal{C}_\mathbb{Q}$ fitting into a commutative diagram

$$
\begin{array}{ccc}
BU & \xrightarrow{\ c\ } & \mathcal{C} \\
\rho \downarrow & & \downarrow \mathbf{av}^\infty \\
\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_Q \xrightarrow{\ f\ } \{BU/\mathfrak{S}_\infty\} & \xrightarrow{\ \gamma^\infty\ } & \mathcal{C}_\mathbb{Q} \simeq \prod_{j=1}^\infty K(\mathbb{Q}, 2j).
\end{array}
$$

In the case of $X = \{pt\}$ we observe that the composition $\gamma^\infty \circ f$ represents a cohomology class $\mathbf{R} \in \prod_{j=1}^\infty H^{2j}(\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)); \mathbb{Q})$, whose identification is rather clear. Consider the polynomial with rational coefficients $R_j(Y_1, \ldots, Y_j)$ which expresses the j-th elementary symmetric function e_j as a polynomial $e_j = R_j(p_1, \cdots, p_j)$ on the Newton power functions $p_k, k = 1, \ldots, j$; cf. [Mac79]. Then we have the paper.

Proposition 5.9. *Let $\mathbf{R}_j \in H^{2j}(\prod_{k \geq 1} K(\mathbb{Z}, 2k), \mathbb{Q})$ be the j-th component of $\mathbf{R}$. Then $\mathbf{R}_j = R_j(\mathbf{i}_2, \ldots, \mathbf{i}_{2j})$, where R_j is the universal polynomial in (60) and $\mathbf{i}_{2k}$ is the rational fundamental class (20).*

When X is smooth we then apply the previous discussion to the universal case, obtaining corresponding relations between the Chern classes and Characters; cf. Theorem 5.10 and Corollary 5.11. In an Appendix we discuss generalities of group-completions of morphisms spaces and set-up the appropriate machinery to deal with the constructions developed along the paper.

The invariants studied in this paper, and the relations between them are further studied in the forthcoming paper [CLF].

2. Symmetric Products and Symmetrized Grassmannnians

In this section we present the basic results relating Grassmannians to symmetric products of projective spaces. Since the objects we study fit into various directed systems of algebraic varieties, we first make some general considerations about such systems.

Consider a directed system $\{Y_\lambda\}_{\lambda \in \Lambda}$ of projective algebraic varieties and morphisms. Although the colimit $\varinjlim_\lambda Y_\lambda$ does not exist in the category of algebraic varieties, it still represents a functor from the category of projective varieties to spaces. More precisely, let X be a projective algebraic variety, and let $\{Y_\lambda\}_{\lambda \in \Lambda}$ be a directed system as above. We denote by $\mathfrak{Map}(X, Y_\lambda)$ the space of continuous maps for the analytic topology from X to

Y_λ, endowed with the compact-open topology. The set of algebraic morphisms $\mathfrak{Mor}(X, Y_\lambda)$ can be topologized as a closed subspace of $\mathfrak{Map}(X, Y_\lambda)$.

Definition 2.1. Given X and $\{Y_\lambda\}_{\lambda \in \Lambda}$ as above, define the morphism space from X to $\varinjlim_\lambda Y_\lambda$ as $\mathfrak{Mor}(X, Y_\infty) := \varinjlim_\lambda \mathfrak{Mor}(X, Y_\lambda)$, where the latter colimit is taken in the category of spaces. If $\{X_\mu\}$ and $\{Y_\lambda\}$ are two such directed systems, define $\mathfrak{Mor}(X_\infty, Y_\lambda) := \varprojlim_\mu \mathfrak{Mor}(X_\mu, Y_\infty)$.

Remark 2.2. Let Y_∞^{an} denote $\mathfrak{Mor}(pt, Y_\infty)$, i.e., the topological colimit of the Y_λ's with their analytic topology. Suppose one has an operation $* : Y_\infty^{an} \times Y_\infty^{an} \to Y_\infty^{an}$ which gives Y_∞^{an} the structure of a topological monoid and such that the restrictions $*_{|Y_\lambda \times Y_\mu}$ are induced by compatible morphisms of varieties $Y_\lambda \times Y_\mu \to Y_{\phi(\lambda, \mu)}$, for all λ and μ. Then, for each projective variety X the operation $*$ naturally induces an structure of topological monoid on $\mathfrak{Mor}(X, Y_\infty)$ via pointwise multiplication.

2.1. **Grassmannians and their symmetrization.**

Here we study actions of the symmetric group on the Grassmannians. Let $\mathfrak{S}_q$ be the symmetric group on q letters and let $\mathbb{C}^{\vee n}$ denote the dual of $\mathbb{C}^n$. The permutation representation of $\mathfrak{S}_q$ on $\mathbb{C}^q$ induces a representation on $\mathbb{C}^{\vee n} \otimes \mathbb{C}^q$, where $\mathfrak{S}_q$ acts trivially on $\mathbb{C}^{\vee n}$. We write the canonical basis of $\mathbb{C}^n$ as $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$, and for each $J = \{j_1 < \cdots < j_k\} \subset \{1, \ldots, n\}$ we let

$$\tag{1} e_J \ : \ \mathbb{C}^k \hookrightarrow \mathbb{C}^n$$

be the linear embedding sending $\mathbf{e}_i$ to $\mathbf{e}_{j_i}$, for $i = 1, \ldots, k$, whose image is denoted by $\mathbb{C}^J \subset \mathbb{C}^n$.

Definition 2.3.

a. Given $q, n \in \mathbb{N}$, let $\mathrm{Gr}_n^q = \mathrm{Gr}^q(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)$ denote the Grassmannian of subspaces of codimension q in $\mathbb{C}^{\vee n} \otimes \mathbb{C}^q$ equipped with the base point $\mathbf{l}_q^n = \mathbf{e}_1^\perp \otimes \mathbb{C}^q$, where $\mathbf{e}_1^\perp$ is the annihilator of $\mathbf{e}_1$ in $\mathbb{C}^{\vee n}$. The Grassmannian $\mathrm{Gr}_n^1 = \mathrm{Gr}^1(\mathbb{C}^{\vee n})$ is the projective space $\mathbb{P}(\mathbb{C}^n)$ of 1-dimensional subspaces of $\mathbb{C}^n$.

b. The action of $\mathfrak{S}_q$ on $\mathbb{C}^{\vee n} \otimes \mathbb{C}^q$ induces an algebraic action on Gr_n^q. In particular, the orbit space $\mathrm{Gr}_n^q/\mathfrak{S}_q$ is a projective algebraic variety; cf. [Har92, p. 127]. Let $\rho_q^n : \mathrm{Gr}_n^q \to \mathrm{Gr}_n^q/\mathfrak{S}_q$ denote the corresponding quotient map, which is also an algebraic map. Note that the base point $\mathbf{l}_q^n \in \mathrm{Gr}_n^q$ is fixed under $\mathfrak{S}_q$, and give $\mathrm{Gr}_n^q/\mathfrak{S}_q$ the base point $\ell_q^n = \rho_q^n(\mathbf{l}_q^n)$.

Using the identification $\mathbb{C}^{\vee n} \otimes \mathbb{C}^{q+q'} = (\mathbb{C}^{\vee n} \otimes \mathbb{C}^q) \oplus (\mathbb{C}^{\vee n} \otimes \mathbb{C}^{q'})$, the direct sum of subspaces induces a based algebraic map

$$\text{(2)} \qquad\qquad \oplus \ : \ \text{Gr}_n^q \times \text{Gr}_n^{q'} \to \text{Gr}_n^{q+q'}$$

$$(\ell, \ell') \mapsto \ell \oplus \ell'.$$

Let

$$\text{(3)} \qquad\qquad \iota_{q,q'} \ : \ \mathfrak{S}_q \times \mathfrak{S}_{q'} \to \mathfrak{S}_{q+q'}$$

denote the usual inclusion, where $\mathfrak{S}_q$ permutes the first q letters and $\mathfrak{S}_{q'}$ permutes the last q' letters. The particular inclusion $\iota_{q,1}$ which keeps the last letter fixed is denoted by $\iota_q \ : \ \mathfrak{S}_q \to \mathfrak{S}_{q+1}$.

Proposition 2.4. *The direct sum operation $\oplus$ is equivariant, in the sense that if $\sigma \in \mathfrak{S}_q$ and $\tau \in \mathfrak{S}_{q'}$, then $(\sigma * \ell) \oplus (\tau * \ell') = \iota_{q,q'}(\sigma, \tau) * (\ell \oplus \ell')$. In particular, it induces a based morphism between the respective quotient varieties*

$$\text{(4)} \qquad\qquad \star \ : \ Gr_n^q / \mathfrak{S}_q \times Gr_n^{q'} / \mathfrak{S}_{q'} \to Gr_n^{q+q'} / \mathfrak{S}_{q+q'},$$

making the following diagram commute:

$$
\begin{array}{ccc}
Gr_n^q \times Gr_n^{q'} & \xrightarrow{\ \oplus\ } & Gr_n^{q+q'} \\
{\scriptstyle \rho_q^n \times \rho_{q'}^n} \downarrow & & \downarrow {\scriptstyle \rho_{q+q'}^n} \\
Gr_n^q / \mathfrak{S}_q \times Gr_n^{q'} / \mathfrak{S}_{q'} & \xrightarrow[\ \star\ ]{} & Gr_n^{q+q'} / \mathfrak{S}_{q+q'}.
\end{array}
$$

Furthermore, the induced maps $\star$ are commutative and associative, in the obvious sense.

Proof. Evident. ∎

We now describe classical stabilizations of the Grassmannians, which are equivariant for the symmetric group action, and whose presentation is necessary for our book keeping.

Given integers n and $q \leq k$, denote by

$$\text{(5)} \qquad\qquad \epsilon_n^{q,k} \ : \ \text{Gr}_n^q \to \text{Gr}_n^k$$

the inclusion which sends $\ell \in \text{Gr}_n^q$ to $\ell \oplus \mathbf{l}_{k-q}^n$; cf. (2).

In order to define the second stabilization map, consider $n \leq m$ and $J = \{1, \dots, n\} \subset \{1, \dots, m\}$, and let $e_J^\vee : \mathbb{C}^{\vee m} \to \mathbb{C}^{\vee n}$ be the adjoint of the map e_J, defined in (1). Then, the surjection $e_J^\vee \otimes \mathbf{1} : \mathbb{C}^{\vee m} \otimes \mathbb{C}^q \to \mathbb{C}^{\vee n} \otimes \mathbb{C}^q$ defines a pull-back map

$$\text{(6)} \qquad\qquad s_{n,m}^q \ : \ \text{Gr}_n^q \to \text{Gr}_m^q,$$

by sending $\ell \subset \mathbb{C}^{\vee n} \otimes \mathbb{C}^q$ to $(e_J^\vee \otimes \mathbf{1})^{-1}(\ell) \subset \mathbb{C}^{\vee m} \otimes \mathbb{C}^q$.

Properties 2.5. Let $q \leq k$ and $n \leq m$ be positive integers. Then
a. The maps $\epsilon_n^{q,k}$ and $s_{n,m}^q$ are algebraic embeddings and satisfy:

$$s_{n,m}^k \circ \epsilon_n^{q,k} = \epsilon_m^{q,k} \circ s_{n,m}^q.$$

See diagram (8) below.

b. The maps $\epsilon_n^{q,k}$ and $s_{n,m}^q$ are equivariant in the sense that, if $\sigma \in \mathfrak{S}_q$ then

$$s_{n,m}^q(\sigma * \ell) = \sigma * (s_{n,m}^q(\ell)) \quad \text{and} \quad \epsilon_n^{q,k}(\sigma * \ell) = \iota_{q,k-q}(\sigma, e) * \epsilon_n^{q,k},$$

where $e \in \mathfrak{S}_{k-q}$ is the identity element; cf. (3).

Since the stabilization maps $\epsilon_*^{*,*}$ and $s_{*,*}^*$ are equivariant, they descend to morphisms of the respective quotient varieties,

(7) $\qquad \epsilon_n^{q,k} \; : \; \mathrm{Gr}_n^q/\mathfrak{S}_q \to \mathrm{Gr}_n^k/\mathfrak{S}_k \quad \text{and} \quad s_{n,m}^q \; : \; \mathrm{Gr}_n^q/\mathfrak{S}_q \to \mathrm{Gr}_m^q/\mathfrak{S}_q.$

Definition 2.6. The colimit of the directed system $(\mathrm{Gr}_n^q, \epsilon_n^{q,k}, s_{n,m}^q)$ of algebraic varieties is our model for BU, the classifying space for stable K-theory. This colimit comes with two filtrations by closed subspaces:

$$\cdots \subset BU_{(n)} \subset BU_{(n+1)} \subset \cdots \subset BU$$

and

$$\cdots \subset BU(q) \subset BU(q+1) \subset \cdots \subset BU.$$

The space $BU_{(n)}$ is defined as the colimit $\varinjlim_q \mathrm{Gr}_n^q$, and each inclusion $s_{n,m} \; : \; BU_{(n)} \hookrightarrow BU_{(m)}$ is a cofibration and a homotopy equivalence. In particular, the inclusions $s_m \; : \; BU_{(m)} \hookrightarrow BU$ are homotopy equivalences. The space $BU(q)$ is the classifying space of q-plane bundles, defined as the colimit $\varinjlim_n \mathrm{Gr}_n^q$. The induced maps $\epsilon^{q,k} \; : \; BU(q) \hookrightarrow BU(k)$ are cofibrations which induce isomorphism in cohomology up to order $2q$.

Remark 2.7. After passing to quotients, one also has a directed system $(\mathrm{Gr}_n^q/\mathfrak{S}_q, \epsilon_n^{q,k}, s_{n,m}^q)$ whose colimit, denoted by $BU/\mathfrak{S}_\infty$, is called *the symmetrized BU*. The filtrations of BU described above descend, under the quotient map $\rho \; : \; BU \to BU/\mathfrak{S}_\infty$, to filtrations of $BU/\mathfrak{S}_\infty$ by cofibrations

$$\cdots \subset BU_{(n)}/\mathfrak{S}_\infty \subset BU_{(n+1)}/\mathfrak{S}_\infty \subset \cdots \subset BU/\mathfrak{S}_\infty$$

and

$$\cdots \subset BU(q)/\mathfrak{S}_q \subset BU(q+1)/\mathfrak{S}_{q+1} \subset \cdots \subset BU/\mathfrak{S}_\infty.$$

All this information can be condensed in the following commutative diagram, where the spaces in the bottom row and in the right column are colimits of their respective column and row.

$$
(8) \quad
\begin{array}{ccccc}
\mathrm{Gr}_n^q/\mathfrak{S}_q & \xrightarrow{\ \epsilon_n^{q,k}\ } & \mathrm{Gr}_n^k/\mathfrak{S}_k & \xrightarrow{\ \epsilon_n^k\ } & BU_{(n)}/\mathfrak{S}_\infty \\
\Big\downarrow{\scriptstyle s_{n,m}^q} & & \Big\downarrow{\scriptstyle s_{n,m}^k} & & \Big\downarrow{\scriptstyle s_{n,m}} \\
\mathrm{Gr}_m^q/\mathfrak{S}_q & \xrightarrow{\ \epsilon_{q,k}^m\ } & \mathrm{Gr}_m^k/\mathfrak{S}_k & \xrightarrow{\ \epsilon_m^k\ } & BU_{(m)}/\mathfrak{S}_\infty \\
\Big\downarrow{\scriptstyle s_m^q} & & \Big\downarrow{\scriptstyle s_m^k} & & \Big\downarrow{\scriptstyle s_m} \\
BU(q)/\mathfrak{S}_q & \xrightarrow{\ \epsilon^{q,k}\ } & BU(k)/\mathfrak{S}_k & \xrightarrow{\ \epsilon^k\ } & BU/\mathfrak{S}_\infty
\end{array}
$$

Recall that the (homotopy theoretic) group completion M^+, of a topological monoid M, is the space ΩBM of loops in its classifying space. See Appendix A for further details.

Proposition 2.8.

a: *The operations $\star$, defined in Proposition 2.4, induce the structure of a graded abelian topological monoid on the coproduct $\mathcal{Q}_{(n)} = \amalg_{q\geq 0}\ \mathrm{Gr}_n^q/\mathfrak{S}_q$, for all n. Furthermore, the natural inclusions $\mathcal{Q}_{(n)} \subset \mathcal{Q}_{(m)}$, induced by the maps $s_{n,m}^q$, are abelian monoid morphisms.*

b: *The same structure assembles to make $BU/\mathfrak{S}_\infty$ an abelian topological monoid, in such a way that each $BU_{(n)}/\mathfrak{S}_\infty$ is a closed submonoid.*

c: *The monoid $BU_{(n)}/\mathfrak{S}_\infty$ is homotopy equivalent to the connected component of the group completion of $\mathcal{Q}_{(n)}$. Similarly, the monoid $BU/\mathfrak{S}_\infty$ is homotopy equivalent to the connected component of the group completion of $\mathcal{Q} = \amalg_{q\geq 0}\ BU(q)/\mathfrak{S}_q$.*

d: *The natural projection $\rho : (BU, \oplus) \to (BU/\mathfrak{S}_\infty, \star)$ is a rational homotopy equivalence and a morphism of infinite loop spaces.*

Proof. Assertions **a** and **b** follow from routine verification.

To prove assertion **c**, first recall that the connected component of the group completion ΩBM of a topological monoid M is homotopy equivalent to the colimit $\varinjlim_{\alpha\in\Lambda} M_\alpha$, where the M_α's are connected components of M, and Λ is a collection contained countably infinitely many copies of each element in $\pi_0(M)$; cf. [Fri91]. This argument implies the following lemma.

Lemma 2.9. *Let M be an abelian topological monoid, equipped with a continuous monoid augmentation $\phi : M \to \mathbb{Z}_+$ onto the additive monoid of the non-negative integers, and let M^+ be its group completion; cf. Appendix A. Define $M_d := \Phi^{-1}(d)$ and chose $\mathbf{1} \in M_1$. Then the colimit $M_\infty := \varinjlim\limits_{d} M_d$ of the system $M_d \xrightarrow{+\mathbf{1}} M_{d+1}$ given by translation by $\mathbf{1}$ is also an abelian topological monoid. Furthermore, ϕ induces a continuous monoid morphism $\Phi : M^+ \to \mathbb{Z}$ and $M_0^+ := \Phi^{-1}(0)$ is homotopy equivalent to the group-completion M_∞^+.*

In our case $\pi_0(\mathcal{Q}_{(n)}) = \mathbb{Z}_+ = \pi_0(\mathcal{Q})$, and assertion **c** follows from the above considerations. To prove assertion **d**, we shall use the following result.

Lemma 2.10. *Let G be a finite group acting on a smooth, connected projective variety X, and let $\rho : X \to X/G$ be the quotient map. Then ρ induces an isomorphism $\rho^* : H^*(X/G; \mathbb{Q}) \to H^*(X; \mathbb{Q})^G$, where G denote the invariants of the cohomology of X under the action of G. Furthermore, if X is simply connected and the fixed point set X^G is non-empty, then X/G is simply connected.*

Proof. The first part of the theorem is well-known and follows from standard transfer arguments. Consider a fixed point $x \in X^G$ and denote $\overline{x} = \rho(x) \in X/G$. It follows from [Ver80] that one can find an equivariant triangulation of X in which x is a vertex, and such the quotient $\rho : X \to X/G$ becomes a simplicial map for the quotient triangulation of X/G. In particular, given any n-simplex $\sigma \subset X/G$, there is a simplex $\tilde{\sigma} \subset X$ such that the restricion of ρ to $\tilde{\sigma}$ is a homeomorphism onto σ.

Standard arguments show that any loop in X/G based on $\overline{x}$ is homotopic to a simplicial loop $\gamma : [0,1] \to X/G$. Therefore, there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ such that $\gamma_{|[t_{i-1},t_i]}$ maps $[t_{i-1}, t_i]$ onto a 1-simplex σ_i of the triangulation in such a way that γ becomes a homeomorphism from (t_{i-1}, t_i) onto the open simplex $\overset{\circ}{\sigma}_i$. For each $i = 1, \ldots, n$ let $\tilde{\sigma}_i$ be a lift of σ_i, and let $r_i : [t_{i-1}, t_i] \to X$ be a reparametrization of its characteristic map by the interval $[t_{i-1}, t_i]$.

Since $\rho(r_2(t_1)) = \gamma(t_1) = \rho(r_1(t_1))$ one can find $g_1 \in G$ such that $g_1 * r_2(t_1) = r_1(t_1)$, and hence the path $\tilde{\gamma} : [0, t_2] \to X$ defined as

$$\tilde{\gamma}(t) = \begin{cases} r_1(t), & t \in [0, t_1] \\ g_1 * r_2(t), & t \in [t_1, t_2] \end{cases}$$

is a lifting of $\gamma_{|[0,t_2]}$. One then proceeds inductively to produce a lifting $\tilde{\gamma}$ of γ. Since $\gamma(0) = \gamma(1) = \overline{x}$ and $\rho^{-1}(\overline{x}) = \{x\}$ because x is a fixed point of the G-action, it follows that $\tilde{\gamma}$ is a loop based on x. The result follows. $\square$

Since the action of $\mathfrak{S}_q$ on Gr_n^q is induced by the natural representation $\mathfrak{S}_q \subset GL(q,\mathbb{C})$ of the general linear group on $\mathbb{C}^{\vee n} \otimes \mathbb{C}^q$, one concludes that $\mathfrak{S}_q$ acts trivially on the cohomology of Gr_n^q. The previous lemma then implies that $\rho_n^q : \mathrm{Gr}_n^q \to \mathrm{Gr}_n^q/\mathfrak{S}_q$ induces an isomorphism between the rational cohomology of two simply-connected spaces. Hence, ρ_n^q is a rational homotopy equivalence. It follows that ρ is also a rational homotopy equivalence. Since ρ is compatible with the direct sum operation on BU, one concludes that it is a map of infinite loop spaces, once we give $BU/\mathfrak{S}_\infty$ the infinite loop space structure coming from the abelian topological monoid structure. We leave the details to the reader. $\square$

Results such as Lemma 2.10 appear in the study of discrete transformation groups. See for example [Rat94, Theorem 13.1.7].

Remark 2.11. Using Remark 2.2 one sees that, given a projective algebraic variety X, the morphism space $\mathfrak{Mor}(X, BU/\mathfrak{S}_\infty)$ has the structure of an abelian topological monoid. It then follows that the assignment $X \mapsto \mathfrak{Mor}(X, BU/\mathfrak{S}_\infty)$ defines a contravariant functor from projective varieties to abelian topological monoids. This can be seen as a presheaf of topological monoids on the site of projective algebraic varieties over $\mathbb{C}$.

2.2. Infinite symmetric products of $\mathbb{P}(\mathbb{C}^\infty)$.

Consider the projective space $\mathbb{P}(\mathbb{C}^n) := \mathrm{Gr}^1(\mathbb{C}^{\vee n})$ of lines in $\mathbb{C}^n$, and let $\mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n)) := \mathbb{P}(\mathbb{C}^n)^{\times q}/\mathfrak{S}_q$ be the q-fold symmetric product of $\mathbb{P}(\mathbb{C}^n)$, with natural projection

$$(9) \qquad t_d^n : \mathbb{P}(\mathbb{C}^n)^{\times q} \to \mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n)).$$

The points in $\mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n))$ are denoted by $a_1 + \cdots + a_q$, $a_i \in \mathbb{P}(\mathbb{C}^n)$, and its base point is $q \cdot \mathbf{l}_1^n$; cf. Definition 2.3.

In a similar fashion to Gr_n^q and $\mathrm{Gr}_n^q/\mathfrak{S}_q$, given $q \leq k$ and $n \leq m$ we consider two stabilizing maps

$$(10) \quad i_{q,k}^n : \mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n)) \to \mathrm{SP}_k(\mathbb{P}(\mathbb{C}^n)) \quad \text{and} \quad j_q^{n,m} : \mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n)) \to \mathrm{SP}_q(\mathbb{P}(\mathbb{C}^m)),$$

defined by $i_{q,k}^n(\sigma) = \sigma + (k-q)\mathbf{l}_1^n$, and $j_q^{n,m}(\sigma) = (s_{n,m}^1)_*(\sigma)$. The latter map is the natural map of symmetric products induced by the inclusion $s_{n,m}^1$ of $\mathbb{P}(\mathbb{C}^n)$ as a linear subspace in $\mathbb{P}(\mathbb{C}^m)$, see (6). The maps $i_{q,k}^n$ and $j_q^{n,m}$ are algebraic embeddings which satisfy

$$j_k^{n,m} \circ i_{q,k}^n = i_{q,k}^m \circ j_q^{n,m}.$$

Definition 2.12. The colimit $\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))$ of the directed system $(\mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n)), i_{q,k}^n, j_q^{n,m})$ is the *infinite symmetric product* of $\mathbb{P}(\mathbb{C}^\infty)$. This space has two filtrations by closed subspaces:

$$\cdots \subset \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^n)) \subset \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^{n+1})) \subset \cdots \subset \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))$$

and

$$\cdots \subset \mathrm{SP}_q(\mathbb{P}(\mathbb{C}^\infty)) \subset \mathrm{SP}_{q+1}(\mathbb{P}(\mathbb{C}^\infty)) \subset \cdots \subset \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)).$$

The subspace $\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^n))$ is defined as the colimit $\varinjlim_q \mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n))$, and each induced map $j^{n,m} : \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^n)) \hookrightarrow \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^m))$ is a cofibration which induces an isomorphism on the homotopy groups up to order $2n$. The subspace $\mathrm{SP}_q(\mathbb{P}(\mathbb{C}^\infty))$ is defined as the colimit $\varinjlim_n \mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n))$ and the induced maps $i_{q,k} : \mathrm{SP}_q(\mathbb{P}(\mathbb{C}^\infty)) \hookrightarrow \mathrm{SP}_k(\mathbb{P}(\mathbb{C}^\infty))$ are cofibrations which induce isomorphisms in homology up to dimension q.

Proposition 2.13. *The usual addition operation $+$ between symmetric products assemble to give $SP_\infty(\mathbb{P}(\mathbb{C}^\infty))$ the structure of an abelian topological monoid, in such a way that each $SP_\infty(\mathbb{P}(\mathbb{C}^n))$ is a closed submonoid.*

Observe that the direct sum induces a map

(11) $$F_n^q : \mathbb{P}(\mathbb{C}^n)^{\times q} = \{\mathrm{Gr}_n^1\}^{\times q} \;\to\; \mathrm{Gr}_n^q$$

which is equivariant for the action of the symmetric group $\mathfrak{S}_q$, and hence induces an algebraic map f_n^q between the respective quotients making the following diagram commute.

(12)
$$\begin{array}{ccc}
\mathbb{P}(\mathbb{C}^n)^{\times q} & \xrightarrow{\;F_n^q\;} & \mathrm{Gr}_n^q \\
{\scriptstyle t_d^n}\big\downarrow & & \big\downarrow{\scriptstyle \rho_n^q} \\
\mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n)) & \xrightarrow[\;f_n^q\;]{} & \mathrm{Gr}_n^q/\mathfrak{S}_q
\end{array}$$

Remark 2.14. Let $\phi_i : \mathbb{P}(\mathbb{C}^n)^{\times q} \to \mathbb{P}(\mathbb{C}^n)$ denote the projection onto the i-th factor, and let $\mathcal{O}(1)$ be the hyperplane line bundle over $\mathbb{P}(\mathbb{C}^n)$. It is a simple geometric fact that $(F_q^n)^*(Q_n^q) = \phi_1^*\mathcal{O}(1) \oplus \cdots \oplus \phi_q^*\mathcal{O}(1)$, where Q_n^q denotes the universal quotient q-plane bundle over Gr_n^q.

Proposition 2.15.

a: *The maps f_n^q form a morphism of directed systems of algebraic varieties*

$$f_*^* : (SP_q(\mathbb{P}(\mathbb{C}^n)),\ i_{*,*}^*,\ j_*^{*,*}) \to (Gr_n^q/\mathfrak{S}_q,\ \epsilon_*^{*,*},\ s_{*,*}^*),$$

so that the induced map of colimits $f : SP_\infty(\mathbb{P}(\mathbb{C}^\infty)) \to BU/\mathfrak{S}_\infty$ is a morphism of abelian topological monoids.

b: *The map f preserves both filtrations of $SP_\infty(\mathbb{P}(\mathbb{C}^\infty))$ and $BU/\mathfrak{S}_\infty$. More precisely,*

$$f(SP_\infty(\mathbb{P}(\mathbb{C}^n))) \subset f(BU_{(n)}/\mathfrak{S}_\infty) \quad and \quad f(SP_q(\mathbb{P}(\mathbb{C}^\infty))) \subset f(BU(q)/\mathfrak{S}_q).$$

Furthermore, the restriction $f^q : SP_q(\mathbb{P}(\mathbb{C}^\infty)) \to BU(q)/\mathfrak{S}_q$ is a rational homotopy equivalence for each q. Hence, f is a rational homotopy equivalence.

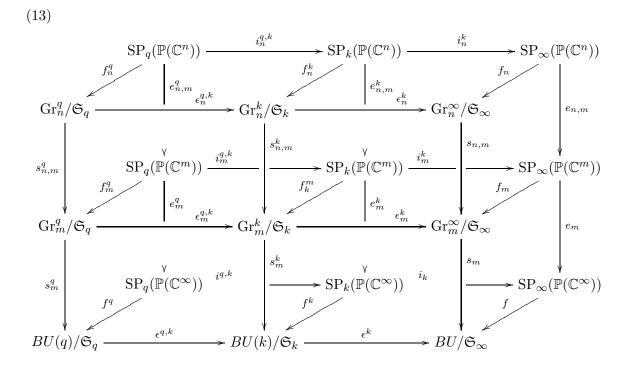*Proof.* Part **a** follows from a simple routine verification. See diagram (13) below.

The filtration preserving property follows from the construction of the maps. It follows from Lemma 2.10 that pull-back under t_q^n gives an isomorphism between $H^*(\mathrm{SP}_d(\mathbb{P}(\mathbb{C}^n)); \mathbb{Q})$ and the invariants $H^*(\mathbb{P}(\mathbb{C}^n)^{\times d}; \mathbb{Q})^{\mathfrak{S}_q}$ under the action of $\mathfrak{S}_q$.

On the other hand, it is well-known that F_q^n also induces an isomorphism $(F_q^n)^* : H^k(\mathrm{Gr}_n^q; \mathbb{Q}) \xrightarrow{\cong} H^k(\mathbb{P}(\mathbb{C}^n)^{\times d}; \mathbb{Q})^{\mathfrak{S}_q}$, whenever $q(n-1) > k$; cf. [MS74]. Using the isomorphisms

$$H^k(\mathrm{Gr}_n^q/\mathfrak{S}_q; \mathbb{Q}) \xrightarrow[(\rho_n^q)^*]{\cong} H^k(\mathrm{Gr}_n^q; \mathbb{Q})^{\mathfrak{S}_q} \cong H^k(\mathrm{Gr}_n^q; \mathbb{Q}),$$

exhibited in the proof of Proposition 2.8, together with $\rho_n^q \circ F_n^q = f_n^q \circ t_q^n$, one concludes that f_n^q induces an isomorphism in the k-th rational cohomology groups, for $q(n-1) > k$. To conclude the proof, first note that a simple inverse limit argument shows that f^q induces an isomorphism in rational cohomology. Then, Lemma 2.10 implies that both $SP_q(\mathbb{P}(\mathbb{C}^\infty))$ and $BU(q)/\mathfrak{S}_q$ are simply-connected, and hence that f^q is a rational homotopy equivalence. $\square$

The following commutative diagram summarizes all the filtrations, colimits and maps involved in Proposition 2.15.

(13)

$$
\begin{array}{ccccc}
\mathrm{SP}_q(\mathbb{P}(\mathbb{C}^n)) & \xrightarrow{\;i_n^{q,k}\;} & \mathrm{SP}_k(\mathbb{P}(\mathbb{C}^n)) & \xrightarrow{\;i_n^k\;} & \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^n)) \\
\end{array}
$$

f_n^q $\quad$ $e_{n,m}^q$ $\quad$ $\epsilon_n^{q,k}$ $\quad$ f_n^k $\quad$ $e_{n,m}^k$ $\quad$ ϵ_n^k $\quad$ f_n

$$
\mathrm{Gr}_n^q/\mathfrak{S}_q \xrightarrow{\;\;} \mathrm{Gr}_n^k/\mathfrak{S}_k \xrightarrow{\;\;} \mathrm{Gr}_n^\infty/\mathfrak{S}_\infty
$$

$s_{n,m}^q$ $\quad$ $s_{n,m}^k$ $\quad$ $s_{n,m}$ $\quad$ $e_{n,m}$

$$
\mathrm{SP}_q(\mathbb{P}(\mathbb{C}^m)) \xrightarrow{\;i_m^{q,k}\;} \mathrm{SP}_k(\mathbb{P}(\mathbb{C}^m)) \xrightarrow{\;i_m^k\;} \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^m))
$$

f_m^q $\quad$ e_m^q $\quad$ $\epsilon_m^{q,k}$ $\quad$ f_k^m $\quad$ e_m^k $\quad$ ϵ_m^k $\quad$ f_m

$$
\mathrm{Gr}_m^q/\mathfrak{S}_q \xrightarrow{\;\;} \mathrm{Gr}_m^k/\mathfrak{S}_k \xrightarrow{\;\;} \mathrm{Gr}_m^\infty/\mathfrak{S}_\infty
$$

s_m^q $\quad$ s_m^k $\quad$ s_m $\quad$ e_m

$$
\mathrm{SP}_q(\mathbb{P}(\mathbb{C}^\infty)) \xrightarrow{\;i^{q,k}\;} \mathrm{SP}_k(\mathbb{P}(\mathbb{C}^\infty)) \xrightarrow{\;i_k\;} \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))
$$

f^q $\quad$ $\epsilon^{q,k}$ $\quad$ f^k $\quad$ ϵ^k $\quad$ f

$$
BU(q)/\mathfrak{S}_q \xrightarrow{\;\epsilon^{q,k}\;} BU(k)/\mathfrak{S}_k \xrightarrow{\;\epsilon^k\;} BU/\mathfrak{S}_\infty
$$

Remark 2.16. Using the same arguments of Remark 2.11 one sees that given a projective variety X, the morphism space $\mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))$ is an abelian topological monoid. Furthermore, the map $f : \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)) \to BU/\mathfrak{S}_\infty$ induces a morphism of abelian topological monoids $f_* : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))) \to \mathfrak{Mor}(X, BU/\mathfrak{S}_\infty)$.

3. RATIONALIZATIONS AND THE CHERN CHARACTER

In this section we present a simple and concrete realization of the Chern character in the level of classifying spaces, and explain the algebraic geometric nature of this realization. We first describe come general facts concerning the localization at 0 of certain specific class of monoids, a process which we call the *rationalization* of the monoid.

Consider an abelian topological monoid M, whose topology is given by an increasing family $\cdots \subset M_n \subset M_{n+1} \subset \cdots \subset M$ of *closed submonoids*. Given $n \leq m$, let $i_{n,m} : M_n \hookrightarrow M_m$ denote the inclusion homomorphism, and define

(14)
$$
\lambda_{n,m} : M_n \to M_m
$$
$$
\sigma \mapsto (\frac{m!}{n!})\, i_{n,m}(\ \sigma\ ).
$$

Observe that the map $r_n : M_n \to M_n$, given by multiplication by $n!$, makes the following diagram commute

(15)
$$
\begin{array}{ccc}
M_n & \xrightarrow{\ r_n\ } & M_n \\
{\scriptstyle i_{n,m}}\downarrow & & \downarrow{\scriptstyle \lambda_{n,m}} \\
M_m & \xrightarrow[\ r_m\ ]{} & M_m.
\end{array}
$$

It follows that the colimit $M_{\mathbb{Q}}$ of the system $\{M_n,\ \lambda_{n,m}\}$, has a natural structure of abelian topological monoid, so that the map $r_M \ :\ M \to M_{\mathbb{Q}}$ induced by the r_n's is a continuous monoid morphism.

Proposition 3.1. *Let M be an abelian topological monoid with a filtration as above, where each $i_{n,m}$ is a cofibration. Then $M_{\mathbb{Q}}$ is a $\mathbb{Q}$-local abelian topological monoid, in that multiplication by any integer $\times m : M_{\mathbb{Q}} \to M_{\mathbb{Q}}$ is invertible. Furthermore, the morphism $r_M \ :\ M \to M_{\mathbb{Q}}$ is a rational homotopy equivalence which represents the localization at 0 of the space M in the homotopy category.*

Proof. It is clear that r_M induces an isomorphism $r_{M*} \ :\ \pi_k(M) \otimes \mathbb{Q} \to \pi_k(M_{\mathbb{Q}})$, for all k. Since an abelian topological monoid is simple, the result follows. $\square$

Remark 3.2. Consider the case where each member of the family $\{\ M_n\ \}$ has the property described in Remark 2.2. In other words, that each M_n has the form $Y_{n,\infty}^{an}$, where $\{Y_{n,\lambda}\}_\lambda$ is a directed system of projective varieties. Then the rationalization $M_{\mathbb{Q}}$ has the structure of a colimit of algebraic varieties and the following properties hold.

1. If X is a projective variety, and M satisfies the property above, then $\mathfrak{Mor}(X, M)$ is a well-defined topological monoid (cf. Definition 2.1) whose topology is given by the increasing sequence of closed submonoids $\cdots \subset \mathfrak{Mor}(X, M_n) \subset \mathfrak{Mor}(X, M_{n+1}) \subset \cdots$.
2. Let $f \ :\ M \to N$ be a monoid morphism where both M and N satisfy the hypothesis of the proposition above. If f is filtration preserving, i.e. $f(M_n) \subset N_n$ for all n, then it induces a monoid morphism $f_{\mathbb{Q}} \ :\ M_{\mathbb{Q}} \to N_{\mathbb{Q}}$ so that $f_{\mathbb{Q}} \circ r_N = r_M \circ f$.

If X is a projective algebraic variety, and M satisfies the property described in the remark above, then $\mathfrak{Mor}(X, M_{\mathbb{Q}})$ is well-defined, cf. Definition 2.1, and it is identified with $\mathfrak{Mor}(X, M)_{\mathbb{Q}}$.

Corollary 3.3. *Let X be a projective algebraic variety, and let M satisfy the property described in Remark 3.2. Then the group completions $\mathfrak{Mor}(X, M_{\mathbb{Q}})^+$ and $(\mathfrak{Mor}(X, M)_{\mathbb{Q}})^+$ coincide.*

Let us recall the canonical splitting of $\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^n))$ introduced by Steenrod and subsequently used by Friedlander and Lawson in [FL92].

The constructions rely on the classical identification $\mathbb{P}^n = \mathrm{SP}_n(\mathbb{P}^1)$. First, choose $x_0 = [1:0] \in \mathbb{P}^1$ as a basepoint. Then for $n \leq q$, the canonical coordinate plane inclusion $\mathbb{P}^n \subset \mathbb{P}^q$ can be identified with the map

$$(16) \qquad i_{n,q} : \mathbb{P}^n = \mathrm{SP}_n(\mathbb{P}^1) \longrightarrow \mathbb{P}^q = \mathrm{SP}_q(\mathbb{P}^1)$$

$$\sigma \longmapsto \sigma + (q-n)x_0.$$

Now, given $n \leq q$, define a morphism $r_{q,n} : \mathbb{P}^q \to \mathrm{SP}_{\binom{q}{n}}(\mathbb{P}^n)$ by sending $a_1 + \cdots + a_q \in \mathrm{SP}_q(\mathbb{P}^1) \equiv \mathbb{P}^q$ to $\sum_{|I|=n}\{a_{i_1} + \cdots + a_{i_n}\} \in \mathrm{SP}_{\binom{q}{n}}(\mathrm{SP}_n(\mathbb{P}^1)) \equiv \mathrm{SP}_{\binom{q}{n}}(\mathbb{P}^n)$. This morphism, in turn, naturally induces a map

$$\mathbf{r}_{q,n} : \mathrm{SP}_\infty(\mathbb{P}^q) \to \mathrm{SP}_\infty(\mathrm{SP}_{\binom{q}{n}}(\mathbb{P}^n)).$$

One has an evident "trace map" (see [FL92, Proposition 7.1]) $tr : \mathrm{SP}_\infty(\mathrm{SP}_{\binom{q}{n}}(\mathbb{P}^n)) \to \mathrm{SP}_\infty(\mathbb{P}^n)$ defined as the extension to the free monoid $\mathrm{SP}_\infty(\mathrm{SP}_{\binom{q}{n}}(\mathbb{P}^n))$ of the natural inclusion $\mathrm{SP}_{\binom{q}{n}}(\mathbb{P}^n) \hookrightarrow \mathrm{SP}_\infty(\mathbb{P}^n)$. This map can be used to define a monoid morphism

$$\rho_{q,n} : \mathrm{SP}_\infty(\mathbb{P}^q) \to \mathrm{SP}_\infty(\mathbb{P}^n)$$

as the composition $\mathrm{SP}_\infty(\mathbb{P}^q) \xrightarrow{\mathbf{r}_{q,n}} \mathrm{SP}_\infty(\mathrm{SP}_{\binom{q}{n}}(\mathbb{P}^n)) \xrightarrow{tr} \mathrm{SP}_\infty(\mathbb{P}^n)$. Note that $\rho_{n,n}$ is the identity map.

Given a projective algebraic variety X, the maps $i_{q,n}$ and $\rho_{q,r}$ induce monoid morphisms $i_{q,n_*} : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^n)) \to \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^q))$ and $\rho_{q,n_*} : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^q)) \to \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^n))$, which in turn induce maps between their respective group-completions

$$i_{q,n_*} : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^n))^+ \to \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^q))^+$$

and

$$\rho_{q,n_*} : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^q))^+ \to \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^n))^+.$$

Definition 3.4. Given a projective variety X define

$$\mathfrak{Mor}(X, \mathrm{SP}_\infty(S^{2n}))^+ := \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^n))^+ /\!\!/ \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^{n-1}))^+,$$

for $n \geq 1$, where the latter denotes the homotopy quotient.

Denote by $\psi_n : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^n))^+ \to \mathfrak{Mor}(X, \mathrm{SP}_\infty(S^{2n}))^+$ the natural homotopy quotient map, and let

$$q_{q,n} : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^q))^+ \to \mathfrak{Mor}(X, \mathrm{SP}_\infty(S^{2n}))^+$$

denote the composition $\psi_n \circ \rho_{q,n_*}$.

It is shown in [FL92] that the map

$$(17) \qquad \Psi^q \; : \; \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}^q))^+ \longrightarrow \prod_{j=1}^{q} \mathfrak{Mor}(X, \mathrm{SP}_\infty(S^{2j}))^+$$

defined as $\Psi^q := \prod_{j=1}^{q} q_{j,q}$, is a homotopy equivalence.

Definition 3.5. Let X be an algebraic variety. The colimit of the maps Ψ^q is denoted by

$$sp^X \; : \; \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+ \longrightarrow \prod_{j=1}^{\infty} \mathfrak{Mor}(X, \mathrm{SP}_\infty(S^{2j}))^+.$$

This map is a homotopy equivalence and is functorial on X.

3.1. **The case** $X = \{pt\}$. Considering $X = pt$ in the discussion above, one obtains the canonical splitting of $\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^n))$

$$(18) \qquad sp_n \; : \; \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^n)) \to \prod_{1 \leq k \leq n-1} \mathrm{SP}_\infty(S^{2k}),$$

since the monoids in question are already group-complete. In this case, the maps sp_n are homotopy equivalences which are also monoid morphisms compatible with both inclusions $\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^n)) \subset \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^{n+1}))$ and $\prod_{1 \leq k \leq n-1} \mathrm{SP}_\infty(S^{2k}) \subset \prod_{1 \leq k \leq n} \mathrm{SP}_\infty(S^{2k})$. Therefore, they induce a canonical filtration-preserving splitting homomorphism

$$(19) \qquad sp \; : \; \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)) \to \prod_{k \geq 1} \mathrm{SP}_\infty(S^{2k}).$$

We use $\mathrm{SP}_\infty(S^{2j})$ as our model for the Eilenberg-MacLane space $K(\mathbb{Z}, 2j)$ (cf. [DT56]), and denote by $\imath_{2j} \in H^{2j}(\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)); \mathbb{Z})$ the class represented by the composition

$$(20) \qquad \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)) \xrightarrow{sp} \prod_{k \geq 1} \mathrm{SP}_\infty(S^{2k}) \xrightarrow{pr_j} \mathrm{SP}_\infty(S^{2j}),$$

where pr_j is the projection. Let $\mathbf{i}_{2j} \in H^{2j}(\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)); \mathbb{Q})$ denote the image of $\imath_{2j}$ under the coefficient homomorphism $\epsilon_* \; : \; H^{2j}(\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)); \mathbb{Z}) \to H^{2j}(\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)); \mathbb{Q})$ induced by the canonical inclusion $\epsilon \; : \; \mathbb{Z} \hookrightarrow \mathbb{Q}$.

Remark 3.6. We use the notation $H^*(X; R)$ to denote the *product* $\prod_{j \geq 1} H^j(X; R)$, for any coefficient ring R.

Theorem 3.7. *Let* $f : SP_\infty(\mathbb{P}(\mathbb{C}^\infty)) \to BU/\mathfrak{S}_\infty$ *be the homomorphism of Proposition 2.15, and identify* $H^*(BU/\mathfrak{S}_\infty; \mathbb{Q})$ *with* $H^*(BU; \mathbb{Q})$ *via the projection* $\rho : BU \to BU/\mathfrak{S}_\infty$. *Then, for* $j \geq 1$ *one has*

$$f^*(\,j!\,\mathbf{ch}_j\,) = \mathbf{i}_{2j},$$

where $\mathbf{ch}_j$ *is the* $2j$-*th component of the total Chern character* $\mathbf{ch} \in H^*(BU; \mathbb{Q})$.

Proof. Let $\jmath_q^n : SP_q(\mathbb{P}(\mathbb{C}^n)) \to SP_\infty(\mathbb{P}(\mathbb{C}^\infty))$ denote the natural map, defined as the composition $e^n \circ i_q^n$; cf. diagram (13). It suffices to show that $(\jmath_q^n)^*(\mathbf{i}_{2j}) = (\jmath_q^n)^*(f^*(j!\mathbf{ch}_j))$ for all $q, n \geq j+1$. Since $t_q^n : \mathbb{P}(\mathbb{C}^n) \times \cdots \times \mathbb{P}(\mathbb{C}^n) \to SP_q(\mathbb{P}(\mathbb{C}^n))$ induces an injection in rational cohomology, we will then show that $(t_q^n)^* \circ (\jmath_q^n)^*(\mathbf{i}_{2j}) = (t_q^n)^* \circ (\jmath_q^n)^* \circ f^*(j!\mathbf{ch}_j)$, for all $q, n \geq j+1$.

Let $x = c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}(\mathbb{C}^n); \mathbb{Q})$ be the generator of the cohomology ring of $\mathbb{P}(\mathbb{C}^n)$, and let $\mathbf{h}_r = [\mathbb{P}(\mathbb{C}^{r+1})] \in H_{2r}(\mathbb{P}(\mathbb{C}^n); \mathbb{Q})$, $r = 1, \ldots, n-1$, be the fundamental class of a coordinate r-plane, the Kronecker dual to x^r. Define $x_i \in H^2(\mathbb{P}(\mathbb{C}^n)^{\times q}; \mathbb{Q})$ as $x_i = \phi_i^*(x)$, where $\phi_i : \mathbb{P}(\mathbb{C}^n)^{\times q} \to \mathbb{P}(\mathbb{C}^n)$ denotes the i-th projection. Given a partition $r_1 + \cdots + r_q = j$ where $j \leq n-1$, let $\mathbf{h}_{r_1} \otimes \cdots \otimes \mathbf{h}_{r_q}$ be the associated generator of $H_{2j}(\mathbb{P}(\mathbb{C}^n)^{\times q}; \mathbb{Q})$, dual to $x_1^{r_1} \cdots x_q^{r_q}$.

Given integers $j \leq n$, one has a commutative diagram:

$$
\begin{array}{ccccc}
\mathbb{P}(\mathbb{C}^j)^{\times q} & \xrightarrow{\ t\ } & SP_q(\mathbb{P}(\mathbb{C}^j)) & \xrightarrow{\ i\ } & SP_\infty(\mathbb{P}(\mathbb{C}^j)) \cong \prod_{s=1}^{j-1} SP_\infty(S^{2s}) \\
\downarrow{\scriptstyle T} & & \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e} \\
\mathbb{P}(\mathbb{C}^n)^{\times q} & \xrightarrow[\ t'\ ]{} & SP_q(\mathbb{P}(\mathbb{C}^n)) & \xrightarrow[\ i'\ ]{} & SP_\infty(\mathbb{P}(\mathbb{C}^n)) \cong \prod_{s=1}^{n-1} SP_\infty(S^{2s}) \xrightarrow[\ p\ ]{} SP_\infty(S^{2j}),
\end{array}
$$

where $t = t_q^j$, $t' = t_q^n$, $i = i_q^j$, $i' = i_q^n$, $e' = e_q^{j,n}$, $e = e^{j,n}$, following the notation of diagram (13), and where T denotes the natural inclusion, and p is the projection.

Consider a partition $r_1 + \cdots + r_q = j$. If some r_i is strictly less than j, then $\mathbf{h}_{r_1} \otimes \cdots \otimes \mathbf{h}_{r_q} = T_*(\varphi)$, where $\varphi \in H_{2j}(\mathbb{P}(\mathbb{C}^j)^{\times q}; \mathbb{Q})$. In this case, if ι_{2j} is the canonical class of $SP_\infty(S^{2j}) = K(\mathbb{Z}, 2j)$, and $\langle \cdot, \cdot \rangle$ denotes the Kronecker pairing, then

$$
\begin{aligned}
\langle \mathbf{h}_{r_1} \otimes \cdots \otimes \mathbf{h}_{r_q}, \ t'^* i'^* p^*(\iota_{2j}) \rangle &= \langle T_*(\varphi), \ t'^* i'^* p^*(\iota_{2j}) \rangle \\
&= \langle p_* i'_* t'_* T_*(\varphi), \ \iota_{2j} \rangle = \langle p_* e_* i_* t_*(\varphi), \ \iota_{2j} \rangle \\
&= 0,
\end{aligned}
$$

(21)

where the last equality follows from the fact that $p \circ e = *$. On the other hand, it follows from the construction of the splittings sp_n that $\langle p_* i'_* t'_*(L_0 \otimes \cdots \otimes L_k \otimes \cdots \otimes L_0), \iota_{2j} \rangle = 1$,

and hence

$$(22) \qquad (t_q^n)^* \circ (j_q^n)^*(\mathbf{i}_{2j}) = x_1^j + \cdots + x_q^j.$$

By definition,

$$(23) \qquad x_1^j + \cdots + x_q^j = \phi_1^*(x)^j + \cdots + \phi_q^*(x)^j = c_1(\phi_1^*\mathcal{O}(1)))^j + \cdots + c_1(\phi_q^*(\mathcal{O}(1)))^j$$
$$= j! ch_j \left(\phi_1^*(\mathcal{O}(1)) \oplus \cdots \oplus \phi_q^*(\mathcal{O}(1)) \right).$$

It follows from (2.14) that

$$(24) \qquad \phi_1^*(\mathcal{O}(1)) \oplus \cdots \oplus \phi_q^*(\mathcal{O}(1)) = (F_n^q)^*(Q_n^q),$$

where Q_n^q is the universal quotient q-plane bundle over G_n^q. Combining (22), (23) and (24), one gets

$$(25) \qquad (t_q^n)^*(j_q^n)^*(\mathbf{i}_{2j}) = j! ch_j((F_n^q)^*(Q^q)) = j!(F_n^q)^*(ch_j(Q_n^q)).$$

Write $ch_j(Q_n^q) = (\rho_n^q)^*(\epsilon_n^q)^* s_n^*(\mathbf{ch}_j)$. Chasing diagram (13) one obtains

$$(26) \qquad j!(F_n^q)^*(ch_j(Q_n^q)) = j!(F_n^q)^*(\rho_n^q)^*(\epsilon_n^q)^* s_n^*(\mathbf{ch}_j))$$
$$= j!(t_q^n)^*(f_n^q)^*(\epsilon_n^q)^* s_n^*(\mathbf{ch}_j)$$
$$= j!(t_q^n)^*(i_q^n)^* e_n^* f^*(\mathbf{ch}_j)$$
$$= (t_n^q)^*(j_q^n)^* f^*(j!\mathbf{ch}_j).$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

In order to obtain the actual Chern character, we apply Proposition 3.1 to our specific situation.

First, define $\rho_B : BU \to \{BU/\mathfrak{S}_\infty\}_\mathbb{Q}$ as the composition $BU \xrightarrow{\rho} BU/\mathfrak{S}_\infty \xrightarrow{r_B} \{BU/\mathfrak{S}_\infty\}_\mathbb{Q}$, where r_B is the rationalization map described in Proposition 3.1. Since ρ is a rational homotopy equivalence, according to Proposition 2.8, we use $\rho_B : BU \to \{BU/\mathfrak{S}_\infty\}_\mathbb{Q}$ as our model for the rationalization of BU. Then observe that the homomorphisms $\mu_n : \mathrm{SP}_\infty(S^{2n}) \to \mathrm{SP}_\infty(S^{2n})$, which sends $\sigma \in \mathrm{SP}_\infty(S^{2n})$ to $n!\sigma$, induce a filtration preserving endomorphism of the (weak) product
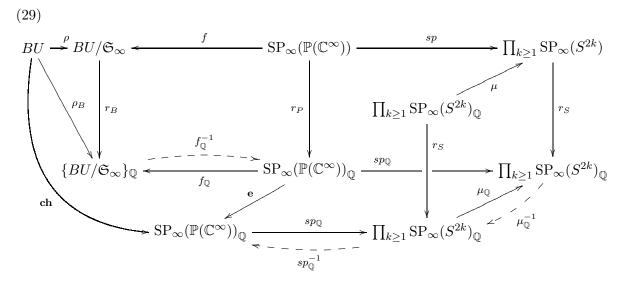
$$(27) \qquad \mu : \prod_{n \geq 1} \mathrm{SP}_\infty(S^{2n}) \to \prod_{n \geq 1} \mathrm{SP}_\infty(S^{2n}).$$

It follows from Remark 3.2 that μ descends to an endomorphism $\mu_\mathbb{Q}$ of the rationalization $\prod_{j \geq 1} \mathrm{SP}_\infty(S^{2j})_\mathbb{Q}$ which is easily seen to be a homotopy equivalence.

Choose homotopy inverses $f_{\mathbb{Q}}^{-1}$, $\mu_{\mathbb{Q}}^{-1}$ and $sp_{\mathbb{Q}}^{-1}$ and define

$$(28) \qquad \mathbf{e} : \mathrm{SP}_{\infty}(\mathbb{P}(\mathbb{C}^{\infty}))_{\mathbb{Q}} \to \mathrm{SP}_{\infty}(\mathbb{P}(\mathbb{C}^{\infty}))_{\mathbb{Q}}$$

as $\mathbf{e} := sp_{\mathbb{Q}}^{-1} \circ \mu_{\mathbb{Q}}^{-1} \circ sp_{\mathbb{Q}}$. These fit into the following diagram whose solid arrows form a commutative diagram, and which becomes homotopy commutative after including the dashed ones.

(29)

$$
\begin{array}{ccccccc}
BU & \xrightarrow{\rho} & BU/\mathfrak{S}_{\infty} & \xleftarrow{f} & \mathrm{SP}_{\infty}(\mathbb{P}(\mathbb{C}^{\infty})) & \xrightarrow{sp} & \prod_{k\geq 1} \mathrm{SP}_{\infty}(S^{2k}) \\
 & \rho_B \searrow & \downarrow r_B & & \downarrow r_P & \mu \nearrow & \downarrow r_S \\
 & & & & \prod_{k\geq 1}\mathrm{SP}_{\infty}(S^{2k})_{\mathbb{Q}} & & \\
 & & \{BU/\mathfrak{S}_{\infty}\}_{\mathbb{Q}} & \xleftarrow[f_{\mathbb{Q}}]{f_{\mathbb{Q}}^{-1}} & \mathrm{SP}_{\infty}(\mathbb{P}(\mathbb{C}^{\infty}))_{\mathbb{Q}} & \xrightarrow{sp_{\mathbb{Q}}} \;\; r_S \;\; & \prod_{k\geq 1}\mathrm{SP}_{\infty}(S^{2k})_{\mathbb{Q}} \\
\mathbf{ch} & & & \mathbf{e} \swarrow & & \mu_{\mathbb{Q}} \nearrow & \mu_{\mathbb{Q}}^{-1} \\
 & & \mathrm{SP}_{\infty}(\mathbb{P}(\mathbb{C}^{\infty}))_{\mathbb{Q}} & \xrightarrow[sp_{\mathbb{Q}}^{-1}]{sp_{\mathbb{Q}}} & \prod_{k\geq 1}\mathrm{SP}_{\infty}(S^{2k})_{\mathbb{Q}} & &
\end{array}
$$

The following result is a simple corollary of the previous constructions and is the desired presentation of the Chern character.

Theorem 3.8. *Let* $\mathbf{ch} : BU \to SP_{\infty}(\mathbb{P}(\mathbb{C}^{\infty}))$ *be the composition* $\mathbf{ch} := \mathbf{e} \circ f_{\mathbb{Q}}^{-1} \circ \rho_B$. *Then* $\mathbf{ch}$ *represents the Chern character. In other words,* $\mathbf{ch}^*(\mathbf{i}_{2j}) = \mathbf{ch}_{2j} \in H^{2j}(BU;\mathbb{Q})$.

Proof. Just observe that the construction of μ implies that $\mu^*(\imath_{2j}) = j!\imath_{2j}$, and hence $\mathbf{e}^*(\mathbf{i}_{2j}) = \frac{1}{j!}\mathbf{i}_{2j}$. The result now follows from Theorem 3.7. $\qquad\square$

3.2. **The case X arbitrary.** We now describe the algebraic-geometric nature of our construction of the Chern character. More precisely, we show that the homotopy inverses $f_{\mathbb{Q}}^{-1}$, $\mu_{\mathbb{Q}}^{-1}$ and sp^{-1} yield uniquely determined homotopy class of maps between group-completed morphism spaces. Observe that, if M is any space in Diagram 29 then it represents a functor $X \mapsto \mathfrak{Mor}(X, M)^+$ from the category of varieties to the category of spaces, as one sees directly from Definition 2.1 and Remark 2.2, using the functoriality of the group completion functor. We will show that the dashed arrows induce natural transformations between the corresponding functors, after passage to the homotopy category.

Consider an algebraic variety X. The splitting map sp^X, introduced in Definition 3.5, is a filtration preserving morphism of topological monoids. Therefore, it induces a natural monoid morphism

$$(30) \qquad sp^X_{\mathbb{Q}} \; : \; \mathcal{M}\mathfrak{or}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+_{\mathbb{Q}} \to \prod_{j \geq 1} \mathcal{M}\mathfrak{or}(X, \mathrm{SP}_\infty(S^{2j}))^+_{\mathbb{Q}},$$

according to Remark 3.2(2). Note that this is still a homotopy equivalence.

Now, let

$$(31) \qquad \mu^X \; : \; \prod_{j \geq 1} \mathcal{M}\mathfrak{or}(X, \mathrm{SP}_\infty(S^{2j}))^+ \to \prod_{j \geq 1} \mathcal{M}\mathfrak{or}(X, \mathrm{SP}_\infty(S^{2j}))^+$$

denote the map induced by the map μ, defined in (27). In other words, $\mu^X(\{f_j\}) = \{j! f_j\}$. In a similar fashion to the case $X = pt$, one sees that μ^X induces a filtration preserving endomorphism of $\prod_{j \geq 1} \mathcal{M}\mathfrak{or}(X, \mathrm{SP}_\infty(S^{2j}))^+$, and hence it induces an endomorphism of the rationalized monoid

$$(32) \qquad \mu^X_{\mathbb{Q}} \; : \; \prod_{j \geq 1} \mathcal{M}\mathfrak{or}(X, \mathrm{SP}_\infty(S^{2j}))^+_{\mathbb{Q}} \to \prod_{j \geq 1} \mathcal{M}\mathfrak{or}(X, \mathrm{SP}_\infty(S^{2j}))^+_{\mathbb{Q}}.$$

Since the monoid is 0-local, this is a homotopy equivalence which is natural on X.

Definition 3.9. Given an algebraic variety X, define

$$\mathbf{e}^X \; : \; \mathcal{M}\mathfrak{or}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+_{\mathbb{Q}} \to \; \mathcal{M}\mathfrak{or}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+_{\mathbb{Q}}$$

as the unique homotopy class of maps given by $\mathbf{e}^X = \left(sp^X_{\mathbb{Q}} \right)^{-1} \circ \left(\mu^X_{\mathbb{Q}} \right)^{-1} \circ sp^X_{\mathbb{Q}}$.

Consider a (smooth) generalized flag variety F, i.e. a compact homogeneous space of the form $F = G/P$, where is a complex algebraic group and $P < G$ is a parabolic subgroup. It follows from the duality results in [FL92] and the computations in [LF92] that the forgetful functor

$$\mathcal{M}\mathfrak{or}(F, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+ \to \; \mathcal{M}\mathfrak{ap}(F, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+$$

is a homotopy equivalence. As a consequence one has the following.

Proposition 3.10. *If G is a finite group of automorphisms of a generalized flag variety F, then the forgetful functor $\mathcal{M}\mathfrak{or}(F/G, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+_{\mathbb{Q}} \to \; \mathcal{M}\mathfrak{ap}(F/G, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+_{\mathbb{Q}}$ is a homotopy equivalence.*

Proof. The argument is standard. The projection $F \to F/G$ gives a morphism $F/G \to \mathrm{SP}_{|G|}(F)$ which in turn it induces a "transfer map"

$$\mathcal{M}\mathfrak{or}(F, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+_{\mathbb{Q}} \to \; \mathcal{M}\mathfrak{or}(F/G, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+_{\mathbb{Q}}$$

which is easily seen to be a homotopy equivalence. The same applies one one replaces $\mathfrak{Mor}(,)$ by $\mathfrak{Map}(,)$ in the construction. The observation preceding the proposition completes the argument. $\square$

Corollary 3.11. *The forgetful maps*

$$\mathfrak{Mor}(\{BU/\mathfrak{S}_\infty\}_\mathbb{Q}, SP_\infty(\mathbb{P}(\mathbb{C}^\infty)))_\mathbb{Q}^+ \to \mathfrak{Map}(\{BU/\mathfrak{S}_\infty\}_\mathbb{Q}, SP_\infty(\mathbb{P}(\mathbb{C}^\infty)))_\mathbb{Q}^+$$

and

$$\mathfrak{Mor}(\{BU/\mathfrak{S}_\infty\}_\mathbb{Q} \times \{BU/\mathfrak{S}_\infty\}_\mathbb{Q}, SP_\infty(\mathbb{P}(\mathbb{C}^\infty)))_\mathbb{Q}^+ \to \mathfrak{Map}(\{BU/\mathfrak{S}_\infty\}_\mathbb{Q} \times \{BU/\mathfrak{S}_\infty\}_\mathbb{Q}, SP_\infty(\mathbb{P}(\mathbb{C}^\infty)))_\mathbb{Q}^+$$

are homotopy equivalences.

Proof. One just needs to observe that these maps are induced by morphisms of inverse systems whose components are maps of the type

$$\mathfrak{Mor}(F/G, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))_\mathbb{Q}^+ \to \mathfrak{Map}(F/G, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))_\mathbb{Q}^+,$$

where F are generalized flag varieties and G is a finite group. Furthermore, all spaces are 0-local. The result follows. $\square$

Throughout the rest of this section we use the notation $B := \{BU/\mathfrak{S}_\infty\}_\mathbb{Q}$ and $S := \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q}$. Let $f_\mathbb{Q} : S \to B$ denote the rationalization of the map introduced in Proposition 2.15. This is a monoid morphism which belongs to $\mathfrak{Mor}(S, B)$. Since $f_\mathbb{Q}$ is also a homotopy equivalence, there is a homotopy inverse $f_\mathbb{Q}^{-1} \in \mathfrak{Map}(B, S)$. The corollary above show that the forgetful functor $\mathfrak{Mor}(B, S)^+ \to \mathfrak{Map}(B, S)^+$ is a homotopy equivalence, and hence there is a unique element $[\alpha] \in \pi_0 \mathfrak{Mor}(B, S)^+$ which maps onto $[f_\mathbb{Q}^{-1}] \in \pi_0 \mathfrak{Map}(B, S)^+$. Let $\alpha \in \mathfrak{Mor}(B, S)^+$ be a representative for $[\alpha]$.

Lemma 3.12. *The element $\alpha \in \mathfrak{Mor}(B, S)^+$ described above is a homotopy homomorphism in the sense of Definition A.4.*

Proof. We know that $f_\mathbb{Q}$ is a monoid morphism, hence $\mu_B \circ (f_\mathbb{Q} \times f_\mathbb{Q}) = f_\mathbb{Q} \circ \mu_S$. Therefore, $f_\mathbb{Q}^{-1} \circ \mu_B \sim \mu_S \circ (f_\mathbb{Q}^{-1} \times f_\mathbb{Q}^{-1})$ as maps. In other words, $[f_\mathbb{Q}^{-1} \circ \mu_B] = [\mu_S \circ (f_\mathbb{Q} \times f_\mathbb{Q})] \in \pi_0(\mathfrak{Map}(B, S)^+)$. This is equivalent to say that

$$(33) \qquad [(f_\mathbb{Q}^{-1})_*^{B \times B}(\mu_B)] = [\mu_{S,+}^{B \times B}(f_\mathbb{Q} \times f_\mathbb{Q})],$$

in the language of Appendix A.

We now use Proposition A.2(b) with $\mathfrak{Mor}(\,,)$ replaced by $\mathfrak{Map}(\,,)$, cf. Remark A.7), to conclude that $\alpha_*^{B \times B}$ is homotopic to $(f_{\mathbb{Q}}^{-1})_*^{B \times B}$. This together with (33) and the definition of α gives the equalities

$$[\alpha_*^{B \times B}(\mu_B)] = [(f_{\mathbb{Q}}^{-1})_*^{B \times B}(\mu_B)] = [\mu_{S,+}^{B \times B}(f_{\mathbb{Q}} \times f_{\mathbb{Q}})] = [\mu_{S,+}^{B \times B}(\alpha \times \alpha)]$$

of elements in $\pi_0 \, \mathfrak{Map}(B \times B, S)^+$. The result now follows from Proposition A.5. $\qquad\square$

We denote a representative for $[\alpha]$ by $f_{\mathbb{Q}}^{-1} \in \mathfrak{Mor}(\{BU/\mathfrak{S}_\infty\}_{\mathbb{Q}}, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+$. It follows from Proposition A.6 that $f_{\mathbb{Q}}^{-1}$ induces an H-space map

(34) $$(f_{\mathbb{Q}}^{-1})_*^X \; : \; \mathfrak{Mor}(X, \{BU/\mathfrak{S}_\infty\}_{\mathbb{Q}}) \to \; \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+,$$

and this assignment is functorial on X.

Theorem 3.13. *The map $(f_{\mathbb{Q}}^{-1})_*^X$ induces a unique homotopy class of maps $(f_{\mathbb{Q}}^{-1})_+^X \; :$ $\mathfrak{Mor}(X, \{BU/\mathfrak{S}_\infty\}_{\mathbb{Q}})^+ \to \; \mathfrak{Mor}(X, SP_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+$, such that $(f_{\mathbb{Q}}^{-1})_+^X \circ u = (f_{\mathbb{Q}}^{-1})_*^X$, where $u : \; \mathfrak{Mor}(X, \{BU/\mathfrak{S}_\infty\}_{\mathbb{Q}}) \to \; \mathfrak{Mor}(X, \{BU/\mathfrak{S}_\infty\}_{\mathbb{Q}})^+$ is the canonical map from the monoid into its group-completion.*

Proof. Denote $M = \; \mathfrak{Mor}(X, \{BU/\mathfrak{S}_\infty\}_{\mathbb{Q}})$, $N = \; \mathfrak{Mor}(X, \{BU/\mathfrak{S}_\infty\}_{\mathbb{Q}})$ and $\alpha = (f_{\mathbb{Q}}^{-1})_*^X$. The multiplicative system $\pi_0(M)$ of the Pontrjagin ring $H_*(M)$ is sent by α to the multiplicative subgroup $\pi_0(N^+)$ of the units of $H_*(N^+)$. Recall that $H_*(M^+)$ is isomorphic to the localization $H_*(M)[\pi_0(M)]^{-1}$; cf. [Q]. Therefore, there is a unique ring homomorphism $\alpha_+ : H_*(M^+) \to H_*(N^+)$ satisfying $\alpha_+ \circ u_* = \alpha_*$. Since both M^+ and N^+ are 0-local abelian topological monoids, they are a product of rational Eilenberg-MacLane spaces and the homomorphism α_+ determines a unique homotopy class of maps satisfying the desired property. $\qquad\square$

Following the same steps as in the case $X = pt$, given an algebraic variety X, we define the map $\mathbf{ch}^X \; : \; \mathfrak{Mor}(X, BU)^+ \to \; \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))_{\mathbb{Q}}^+$ as the composition

(35) $$\mathbf{ch}^X \; = \; \mathbf{e}^X \circ (f_{\mathbb{Q}}^{-1})_*^X \circ \rho_{B,*}^X,$$

where $\mathbf{e}^X$ is introduced in Definition 3.9, and $\rho_{B,*}^X : \; \mathfrak{Mor}(X, BU)^+ \to \; \mathfrak{Mor}(X, \{BU/\mathfrak{S}_\infty\}_{\mathbb{Q}})^+$ is the map induced by the projection ρ_B. See Diagram 29.

Let us explain the significance of these constructions. Given a projective algebraic variety X, the homotopy groups

(36) $$L^j H^{2j-i}(X) := \pi_i(\; \mathfrak{Mor}(X, \mathrm{SP}_\infty(S^{2j}))^+)$$

were introduced in [FL92] and are called the *morphic cohomology* groups of X.

Definition 3.14. The (reduced) holomorphic K-theory space of X is defined as $\tilde{\mathcal{K}}_{hol}(X) :=$ $\mathfrak{Mor}(X, BU)^+$, and the (reduced) *holomorphic K-theory groups* of X are defined as $\tilde{\mathcal{K}}_{hol}^{-i}(X) :=$ $\pi_i(\tilde{\mathcal{K}}_{hol}(X))$.

Remark 3.15.   1. Our holomorphic K-groups coincides with the "semi-topological" K-groups studied in [FW99]. We study holomorphic K-theory in greater generality and in much more depth in [CLF].

2. As shown in [CLF], $\tilde{\mathcal{K}}_{hol}(X)$ is an infinite loop space that corresponds to the zero-th space of the spectrum introduced in [LLFM96] and whose basic properties are discussed in [LF99].

As a main consequence of the constructions in this section, we obtain the following result.

Theorem 3.16. *Let X be a projective variety. The natural maps* $\mathbf{ch}^X$: $\mathfrak{Mor}(X, BU)^+ \to$ $\mathfrak{Mor}(X, SP_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+_\mathbb{Q}$ *induce natural homomorphisms*

$$\mathbf{ch}^i_X : \tilde{\mathcal{K}}_{hol}^{-i}(X) \to \prod_{j \geq 0} L^j H^{2j-i}(X)_\mathbb{Q}$$

from the holomorphic K-theory groups of X to its rational morphic cohomology. These homomorphism fit into a commutative diagram

$$
\begin{array}{ccc}
\tilde{\mathcal{K}}_{hol}^{-i}(X) & \longrightarrow & \tilde{\mathcal{K}}_{top}^{-i}(X) \\
\mathbf{ch}^i_X \downarrow & & \downarrow \mathbf{ch}^i_X \\
\prod_{j \geq 0} L^j H^{2j-i}(X)_\mathbb{Q} & \longrightarrow & \prod_{j \geq 0} H^{2j-i}(X, \mathbb{Q}),
\end{array}
$$

where the right vertical arrow is the usual Chern character from topological K-theory to ordinary cohomology category, cf. Theorem 3.8, and the horizontal arrows are given by the usual forgetful functors.

Proof. The result follows from the naturality of the constructions and the case $X = \{pt\}$. $\square$

Remark 3.17. The lower horizontal arrow in the diagram is the "cycle map" from the morphic cohomology of X to its singular cohomology; cf. [FL92].

Definition 3.18. We call the natural maps

$$\mathbf{ch}^i_X : \tilde{\mathcal{K}}_{hol}^{-i}(X) \to \prod_{j \geq 0} L^j H^{2j-i}(X)_\mathbb{Q}$$

the *Chern character* from the (reduced) holomorphic K-theory of X to its morphic cohomology. Setting $X = BU$ one obtains the *the tautological Chern character element*

$\mathbf{ch} \in \mathfrak{Mor}(BU, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))_\mathbb{Q}^+$ defined as $\mathbf{ch} := \mathbf{ch}^{BU}(Id)$, where $Id \in \mathfrak{Mor}(BU, BU)^+$ is the identity.

Remark 3.19. The space $\mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+$ can be made into a homotopy ring in such a way that the forgetful functor $\mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+ \to \mathfrak{Map}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+$ is a homotopy ring homomorphism when the latter space is given the multiplicative structure corresponding to the cup product. This will be explained in §4.2, after the discussion of the join pairing in Theorem 4.9. We will then see that the tensor product of bundles induces a pairing $\otimes^X : \tilde{\mathcal{K}}_{hol}(X) \wedge \tilde{\mathcal{K}}_{hol}(X) \to \tilde{\mathcal{K}}_{hol}(X)$ which makes the Chern character into a homotopy ring homomorphism; cf. Proposition 4.11.

4. CHOW VARIETIES AND CHERN CLASSES

In this section we consider spaces of algebraic cycles on projective spaces, together with their stabilizations and rationalizations. These spaces are natural recipients for Chern classes and give another algebraic-geometric model for products of Eilenberg-MacLane spaces. Most of the results presented here are an adaptation of results from [BLLF$^+$93], [LLFM96] and [LF99] to the present context.

4.1. **Algebraic cycles on** $\mathbb{P}(\mathbb{C}^\infty)$ **and Chern classes; the case** $X = \{pt\}$.

Definition 4.1. Given $n > 0$ and $q \geq 0$, let $\mathcal{C}_{n,d}^q = \mathrm{Chow}_d^q\big(\mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)\big)$ be the *Chow variety* consisting of the effective algebraic cycles of codimension q and degree d in $\mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)$; cf. [Law95]. The formal *addition* of cycles

$$+ : \mathcal{C}_{n,d}^q \times \mathcal{C}_{n,d'}^q \to \mathcal{C}_{n,d+d'}^q$$

is an algebraic map which makes $\mathcal{C}_{n,*}^q := \amalg_{d \geq 0}\mathcal{C}_{n,d}^q$, into a graded abelian topological monoid, called the *Chow monoid* of effective cycles of codimension q in $\mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)$. This is the free abelian monoid generated by the irreducible subvarieties of codimension q in $\mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)$.

Remark 4.2. Given a complex vector space V, there is a 1-1 correspondence between irreducible subvarieties $Z \subset \mathbb{P}(V)$ and irreducible cones $\mathrm{Cone}(Z) \subset V$. This correspondence identifies the Chow variety $\mathcal{C}_{n,1}^q = \mathrm{Chow}_1^q\big(\mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)\big)$ of cycles of degree one in $\mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)$ with the Grassmannian Gr_n^q. Under this identification one obtains a natural embedding $c_q : \mathrm{Gr}_n^q \hookrightarrow \mathcal{C}_{n,*}^q$ as a connected component.

An important feature of the Chow monoids is the fact that they come equipped with an "exterior" bilinear multiplication

$$(37) \qquad \sharp \; : \; \mathcal{C}_{n,d}^q \times \mathcal{C}_{n,e}^{q'} \; \to \; \mathcal{C}_{n,de}^{q+q'}$$

given by the *ruled join* of cycles; cf. [Law95]. This operation is described as follows. Let $i : \mathbb{C}^{\vee n} \otimes \mathbb{C}^q \to \mathbb{C}^{\vee n} \otimes \mathbb{C}^{q+q'}$ and $j : \mathbb{C}^{\vee n} \otimes \mathbb{C}^{q'} \to \mathbb{C}^{\vee n} \otimes \mathbb{C}^{q+q'}$ be the complementary embeddings induced by the inclusion of $\mathbb{C}^q$ into $\mathbb{C}^{q+q'}$ given by the first q coordinates and of $\mathbb{C}^{q'}$ into $\mathbb{C}^{q+q'}$ as the last q' ones. Consider a pair of subvarieties $Z \subset \mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)$ and $W \subset \mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^{q'})$. One defines the subvariety $Z \sharp W \subset \mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^{q+q'})$ as the union of all projective lines in $\mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^{q+q'})$ which join points in $i(Z)$ to points in $j(W)$. One extends $\sharp$ to arbitrary cycles by linearity.

The join satisfies the following properties:

Facts 4.3.
 a. The join is a strictly associative operation.
b. Its restriction to the connected components yields an algebraic map

$$(38) \qquad \sharp : \mathcal{C}_{n,d}^q \times \mathcal{C}_{n,e}^{q'} \to \mathcal{C}_{n,de}^{q+q'};$$

cf. [Plü97] and [Bar91].
c. In the particular case of cycles of degree one ($d = d' = 1$), the join coincides with the usual direct sum operation

$$(39) \qquad \oplus : \mathrm{Gr}_n^q \times \mathrm{Gr}_n^{q'} \to \mathrm{Gr}_n^{q+q'};$$

cf. Remark 4.2.

We now proceed to introduce three algebraic maps

$$(40) \qquad t_{n,(d,e)}^q \; : \; \mathcal{C}_{n,d}^q \to \mathcal{C}_{n,e}^q,$$

$$(41) \qquad \epsilon_{n,d}^{(q,k)} \; : \; \mathcal{C}_{n,d}^q \to \mathcal{C}_{n,d}^k,$$

$$(42) \qquad s_{(n,m),d}^q \; : \; \mathcal{C}_{n,d}^q \to \mathcal{C}_{m,d}^q,$$

which will define a directed system

$$(43) \qquad \{\mathcal{C}_{n,d}^q; t_{n,(d,e)}^q, \epsilon_{d,n}^{(q,k)}, s_{(n,m),d}^q\}.$$

Given $d \le e$, we define $t_{n,(d,e)}^q : \mathcal{C}_{n,d}^q \to \mathcal{C}_{n,e}^q$ by $t_{n,(d,e)}^q(\sigma) = \sigma + (e-d)\mathbf{l}_q^n$. This uses the additive structure of the Chow monoid. The maps $\epsilon_{n,d}^{(q,k)}$ and $s_{(n,m),d}^q$ are extensions to

higher degrees of the maps $\epsilon_n^{q,k}$ and $s_{n,m}^q$ introduced in (5) and (6), and are defined as follows.

First, given $J \subset \{1, \ldots, k\}$ with $|J| = q$, let J^c denote its complement. Then let $sh_J \in \mathfrak{S}_k$ denote the shuffle permutation which sends the ordered k-tuple (J, J^c) to $(1, \ldots, k)$. Under the permutation representation, sh_J induces an isomorphism $sh_J : \mathbb{C}^k \to \mathbb{C}^k$ and we define $e_J : \mathbb{C}^q \hookrightarrow \mathbb{C}^k$ and $e_{J^c} : \mathbb{C}^{k-q} \hookrightarrow \mathbb{C}^k$ as the compositions $\mathbb{C}^q \xrightarrow{i} \mathbb{C}^k \xrightarrow{sh_J} \mathbb{C}^k$ and $\mathbb{C}^{k-q} \xrightarrow{j} \mathbb{C}^k \xrightarrow{sh_J} \mathbb{C}^k$. These are the maps defined in (1). Define

$$(44) \qquad\qquad \epsilon_{n,d}^J : \mathcal{C}_{n,d}^q \to \mathcal{C}_{n,d}^k$$

by $\epsilon_{n,d}^J(\sigma) = sh_{J*}(\sigma \sharp \mathbf{l}_{k-q}^n)$, where sh_{J*} is the map on cycles induced by the shuffle map; see cf. (37). In the particular case where $J = \{1, \ldots, q\} \subset \{1, \ldots, k\}$ we denote $\epsilon_{n,d}^J$ by $\epsilon_{n,d}^{(q,k)}$.

Remark 4.4. Observe that the maps $e_{n,d}^J$ are all homotopic to $e_{n,d}^{(q,k)}$ for all choices of $J \subset \{1, \ldots, k\}$ with $|J| = q$.
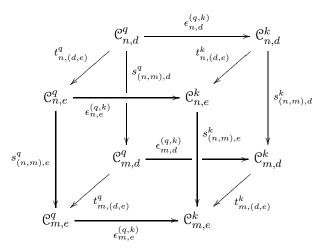
In order to define the third stabilization map, we consider $n \leq m$ and $J = \{1, \ldots, n\} \subset \{1, \ldots, m\}$, and let $e_J^\vee : \mathbb{C}^{\vee m} \to \mathbb{C}^{\vee n}$ be the adjoint of the map e_J, defined in (1). Then, the surjection $e_J^\vee \otimes \mathbf{1} : \mathbb{C}^{\vee m} \otimes \mathbb{C}^q \to \mathbb{C}^{\vee n} \otimes \mathbb{C}^q$ induces a map

$$(45) \qquad\qquad s_{(n,m),d}^q : \mathcal{C}_{n,d}^q \to \mathcal{C}_{m,d}^q,$$

defined as follows. Given an irreducible subvariety $Z \subset \mathbb{P}(\mathbb{C}^{\vee n} \otimes \mathbb{C}^q)$ of degree d, let $s_{(n,m),d}^q(Z) \subset \mathbb{P}(\mathbb{C}^{\vee m} \otimes \mathbb{C}^q)$ be the irreducible variety of whose cone $\text{Cone}(s_{(n,m),d}^q(Z)) \subset \mathbb{C}^{\vee m} \otimes \mathbb{C}^q$ is defined as $(e_J^\vee \otimes \mathbf{1})^{-1}(\text{Cone}(Z))$. Then, extend $s_{(n,m),d}^q$ linearly to arbitrary cycles.

Remark 4.5. One could rephrase the last definition in terms of a suitable join operation, and vice-versa. We prefer this approach, for it is a direct generalization of the Grassmannians case.

Lemma 4.6. *Given* $q \leq k$, $n \leq m$ *and* $d \leq e$, *the following diagram commutes.*

Proof. This is just a careful diagram chase using the definitions. ☐

Definition 4.7. The colimit $\mathcal{C} := \varinjlim_{q,n,d} \mathcal{C}^q_{n,d}$ of the directed system $\{\mathcal{C}^q_{n,d}; t^q_{n,(d,e)}, \epsilon^{(q,k)}_{d,n}, s^q_{(n,m),d}\}$ is the "stabilized" Chow variety of effective cycles cycles in $\mathbb{P}(\mathbb{C}^{\vee\infty} \otimes \mathbb{C}^\infty)$.

Remark 4.8. We may fix some of the parameters q, n, d and let the other(s) go to infinity, obtaining intermediate spaces. The corresponding notation will use the symbol ∞ whenever appropriate, to denote the colimits, and maps between them. For example, by fixing one of the parameters q, n or d, we have the spaces $\mathcal{C}^q_{\infty,\infty}$, $\mathcal{C}^\infty_{n,\infty}$ and $\mathcal{C}^\infty_{\infty,d}$ when the colimit is taken over the remaining two parameters. This gives three filtrations of $\mathcal{C}$ by cofibrations:

$$
\text{(46)} \qquad \ldots \subset \mathcal{C}^\infty_{\infty,d} \subset \mathcal{C}^\infty_{\infty,d+1} \subset \cdots \subset \mathcal{C},
$$

$$
\text{(47)} \qquad \ldots \subset \mathcal{C}^q_{\infty,\infty} \subset \mathcal{C}^{q+1}_{\infty,\infty} \subset \cdots \subset \mathcal{C},
$$

and

$$
\text{(48)} \qquad \ldots \subset \mathcal{C}^\infty_{n,\infty} \subset \mathcal{C}^\infty_{n+1,\infty} \subset \cdots \subset \mathcal{C}.
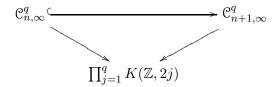$$

We have seen that $\mathcal{C}^q_{n,*} := \amalg_{d\geq 0}\mathcal{C}^q_{n,d}$ is an abelian topological monoid and that $s^q_{(n,m),d}$ and $\epsilon^{(q,k)}_{n,d}$ are both monoid morphisms. Therefore, the colimit $\mathcal{C}^\infty_{\infty,*} := \amalg_{d\geq 0}\mathcal{C}^\infty_{\infty,d}$ is also an abelian topological monoid with a continuous augmentation $\phi : \mathcal{C}^\infty_{\infty,*}$. In [LM88], $\mathcal{C}^\infty_{\infty,d}$ is denoted by $\mathcal{D}_d$, and $\mathcal{C}$ is denoted by $\mathcal{D}_\infty$.

The following theorem summarizes various results proven in [Law89], [LM88], [FL92] and [BLLF⁺93]. The presentation here is chosen to provide a parallel with the analogous results in the previous presentation of symmetric products and Grassmannians.

Theorem 4.9.

a. *Addition of cycles gives* $\mathcal{C}$ *the structure of an abelian topological monoid. With this structure,* $\mathcal{C}$ *is homotopy equivalent to the connected component of the group completion* $(\mathcal{C}_{\infty,*}^{\infty})^{+} = \left(\amalg_{d\geq 0}\mathcal{C}_{\infty,d}^{\infty}\right)^{+}.$

b. *Each monomorphism* $\mathcal{C}_{n,\infty}^{q} \subset \mathcal{C}_{n+1,\infty}^{q}$ *is a homotopy equivalence, and there are compatible canonical splittings* $\mathcal{C}_{n,\infty}^{q} \simeq \prod_{j=1}^{q} K(\mathbb{Z},2j)$, *fitting in a commutative diagram*

$$\mathcal{C}_{n,\infty}^{q} \lhook\joinrel\longrightarrow \mathcal{C}_{n+1,\infty}^{q}$$
$$\searrow \qquad \swarrow$$
$$\prod_{j=1}^{q} K(\mathbb{Z},2j)$$

c. *Each monomorphism* $\mathcal{C}_{n,\infty}^{q} \subset \mathcal{C}_{n,\infty}^{q+1}$ *is homotopic to the inclusion of* $\mathcal{C}_{n,\infty}^{q} \simeq \prod_{j=1}^{q} K(\mathbb{Z},2j)$ *as a factor in* $\mathcal{C}_{n,\infty}^{q+1} \simeq \prod_{j=1}^{q+1} K(\mathbb{Z},2j)$.

d. *The natural inclusions* $Gr_{n}^{q} \equiv \mathcal{C}_{n,1}^{q} \hookrightarrow \mathcal{C}_{n,\infty}^{q} \simeq \prod_{j=1}^{q} K(\mathbb{Z},2j)$ *stabilize as* $n \to \infty$ *to give the truncated total Chern class* $c^{q} : BU(q) \to \mathcal{C}_{\infty,\infty}^{q} \simeq \prod_{j=1}^{q} K(\mathbb{Z},2j)$.

e. *By sending* $q \to \infty$ *one gets the total Chern class* $c : BU \to \mathcal{C} \simeq \prod_{j=1}^{\infty} K(\mathbb{Z},2j)$. *This is a bi-filtration preserving map, satisfying* $c(BU(q)) \subset \mathcal{C}_{\infty,\infty}^{q} \subset \mathcal{C}$ *and* $c(BU_{(n)}) \subset \mathcal{C}_{n,\infty}^{\infty} \subset \mathcal{C}$.

f. *Under the join operation* $\sharp$, *the monoid* $\mathcal{C}_{\infty,*}^{\infty} = \amalg_{d\geq 0}\mathcal{C}_{\infty,d}^{\infty}$ *becomes an* E_{∞}-*ring space. Therefore the group completion* $\left(\mathcal{C}_{\infty,*}^{\infty}\right)^{+}$ *has an* E_{∞}-*ring augmentation induced by the degree of cycles* $\Phi : \left(\mathcal{C}_{\infty,*}^{\infty}\right)^{+} \to \mathbb{Z}$. *Furthermore, the connected component of* 1 *in the group completion* $\left(\mathcal{C}_{\infty,*}^{\infty}\right)_{1}^{+} := \Phi^{-1}(1)$, *which can be identified with* $\mathcal{C}$, *has a "multiplicative" infinite loop space structure for which the total Chern class map* $c : (BU,\oplus) \to (\mathcal{C},\sharp)$ *is a map of infinite loop spaces.*

Proof. Assertion **a** is proven in [Fri91], and the arguments in the proof are outlined in the proof of Proposition 2.8. Assertions **b** and **c** follow from Lawson's complex suspension theorem [Law89] and the splittings of [FL92]. Assertions **d** and **e** follow from [LM88] and a mere inspection of the definitions of the filtrations. The last assertion is proven in [BLLF$^+$93]. □

4.2. The case X arbitrary.

Let X be a projective algebraic variety. The identification $BU(q) = Gr_{\infty}^{q} = \mathcal{C}_{\infty,1}^{q}$ gives an inclusion $BU(q) \hookrightarrow \amalg_{d} \mathcal{C}_{\infty,d}^{q}$, and letting q go to infinity, one gets a map $\mathcal{M}or(X,BU) \to$

$\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})$, where $\mathcal{C}^\infty_{\infty,*} = \amalg_d \mathcal{C}^q_{\infty,d}$. Denote by

(49) $$c_X : \ \mathfrak{Mor}(X, BU) \ \to \ \mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+$$

the composition of the map above with the universal map $\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*}) \to \mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+$ from $\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})$ to its additive group-completion; cf. Appendix A.

In [LLFM96] and [LF99] it is shown that $\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+$ is an abelian topological monoid with a multiplicative action of the *linear isometries operad* $\mathcal{L}$ induced by the join pairing on algebraic cycles. This gives $\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+$ the structure of an augmented E_∞-ring space, in the language of [LLFM96], with augmentation $\Phi : \mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+ \to \mathbb{Z}$. Define $\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_d := \Phi^{-1}(d)$ and recall that Lemma 2.9 gives a homotopy equivalence $\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_0 \cong \mathfrak{Mor}(X, \mathcal{C})^+$. Hence one has natural equivalences

(50) $$\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1 \cong \mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_0 \cong \mathfrak{Mor}(X, \mathcal{C})^+,$$

where the former one is given by translation by the element $\mathbf{1} \in \mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1$, represented by a constant map $X \to \mathcal{C}^\infty_{\infty,1}$.

Remark 4.10. Using the complex suspension theorem of [FL92], one obtains a canonical homotopy equivalence

$$\Sigma^X : \ \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+ \to \ \mathfrak{Mor}(X, \mathcal{C})^+,$$

hence $\mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+$ becomes a homotopy ring space, with mutiplication

$$\# : \mathfrak{Mor}(X, \mathcal{C})^+ \times \ \mathfrak{Mor}(X, \mathcal{C})^+ \to \ \mathfrak{Mor}(X, \mathcal{C})^+$$

induced by the join of cycles. It follows from [LM88] that the forgetful functor

$$\mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+ \to \ \mathfrak{Map}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+$$

is a map of ring spaces where the homotopy ring structure in the later is induced by the cup product.

We make a brief digression here to explore this multiplicative structure further. The tensor product of vector bundles gives an element $\otimes \in \ \mathfrak{Mor}(BU \times BU, BU)^+$, which satisfies the following property. Let $\mathbf{ch} \in \ \mathfrak{Mor}(BU \times BU, BU)$ be the Chern character element, cf. Definition 3.18, and consider the elements $\#^{BU \times BU}_S(\mathbf{ch})$ and $\mathbf{ch}^{BU \times BU}(\otimes) \in \mathfrak{Mor}(BU \times BU, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))^+_\mathbb{Q},)$. The fact that the topological Chern character is a ring homomorphism implies that these elements represent the same element in $\pi_0(\mathfrak{Map}(BU \times BU, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))^+_\mathbb{Q},))$. Then it follows from Corollary 3.11 that they represent the same element in $\pi_0(\mathfrak{Mor}(BU \times BU, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))^+_\mathbb{Q},))$. Another application of Proposition A.6 proves the following.

Proposition 4.11. *Given a projective algebraic variety X, the Chern character map $\mathbf{ch}^X$: $\mathfrak{Mor}(X, BU)^+ \to \mathfrak{Mor}(X, SP_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+_{\mathbb{Q}}$ is a homotopy ring homomorphism. In particular $\mathbf{ch}^0_X : \tilde{\mathcal{K}}^0_{hol}(X) \to \prod_{j \geq 0} L^j H^{2j}(X)_{\mathbb{Q}}$ is a ring homomorphism and $\mathbf{ch}^i_X : \tilde{\mathcal{K}}^{-i}_{hol}(X) \to \prod_{j \geq 0} L^j H^{2j-i}(X)_{\mathbb{Q}}$ is a homomorphism of modules over $\tilde{\mathcal{K}}^0_{hol}(X)$.*

It is easy to see that c_X factors as $\mathfrak{Mor}(X, BU) \to \mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1 \to \mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+$. Furthermore, the map $\mathfrak{Mor}(X, BU) \to \mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1$ is a map of $\mathcal{L}$-spaces, cf. [LF99], and hence it induces a map between their respective group completions

$$(51) \qquad c_X : \mathfrak{Mor}(X, BU)^+ \to \Omega B \left(\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1 \right).$$

Observe that the latter group-completion is taken with respect to the join pairing structure on $\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1$.

In [LF99] it is shown that, for X smooth, $\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1$ is group-complete with respect to the join pairing, hence

$$(52) \qquad \mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1 \cong \Omega B \left(\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1 \right).$$

The argument goes as follows. Consider $\pi_0(\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1) \cong 1 \times \prod_{p \geq 1} L^p H^{2p}(X)$, where $L^* H^*(X)$ denotes the morphic cohomology groups (36). It is shown in [FL92] that whenever X is smooth then $L^p H^{2p}(X) \cong A^{2p}(X)$, where the latter denotes the Chow group of algebraic cycles of codimension p modulo algebraic equivalence. Furthermore, the multiplication on $1 \times \prod_{p \geq 1} L^p H^{2p}(X)$ induced by the join coincides with the intersection pairing. Hence $\pi_0(\mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1)$ is a group under the join multiplication.

Definition 4.12. Let X be a smooth algebraic variety. Using (51) and (52) we can construct the *total Chern class map*

$$\mathbf{c}_X : \mathfrak{Mor}(X, BU)^+ \to \mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1.$$

One can combine the suspension equivalence Σ^X, with the splitting in Definition 3.5 and (36), and then take homotopy groups to define the higher Chern class maps

$$\mathbf{c}^i_X : \tilde{\mathcal{K}}^{-i}_{hol}(X) \to \prod_{p \geq 1} L^p H^{2p-i}(X)$$

from the (reduced) holomorphic K-theory of X to its morphic cohomology. The individual components of this map are denoted by $\mathbf{c}^{p,i}_X : \tilde{\mathcal{K}}^{-i}_{hol}(X) \to L^p H^{2p-i}(X)$.

Remark 4.13.   **a:** The identification (50) allows one to identify $\mathbf{c}_X : \mathfrak{Mor}(X, BU)^+ \to \mathfrak{Mor}(X, \mathcal{C}^\infty_{\infty,*})^+_1$ with a map $\mathbf{c}_X : \mathfrak{Mor}(X, BU)^+ \to \mathfrak{Mor}(X, \mathcal{C})^+$.

b: It is shown in [LF99] that $\mathbf{c}_X$ is a map of spectra from the holomorphic K-theory spectrum of X to its morphic spectrum, in the terminology of [LLFM96].

c: Under the forgetful functor $\mathfrak{Mor}(,) \to \mathfrak{Map}(,)$ one obtains a commutative diagram

$$
\begin{array}{ccc}
\tilde{\mathcal{K}}^{-i}_{hol}(X) & \xrightarrow{\ \mathbf{c}^i_X\ } & \prod_{p \geq 1} L^p H^{2p-i}(X) \\
\downarrow & & \downarrow \\
\tilde{\mathcal{K}}^{-i}_{top}(X) & \xrightarrow{\ \mathbf{c}\ } & \prod_{p \geq 1} H^{2p-i}(X; \mathbb{Z})
\end{array}
$$

where $\tilde{\mathcal{K}}^{-i}_{top}(X)$ is the reduced topological K-theory of X and $\mathbf{c}$ is the usual Chern class map into singular cohomology.

5. Relations between the Chern character and Chern classes

In this section we present a relation between the Chern characters and the Chern classes contructed in the previous sections. This requires an alternative description of the rational Chern class map, and some new constructions with cycle spaces.

5.1. **Rationalization of cycle spaces.** We first describe a modification of the directed system (43), aiming at rationalizing $\mathcal{C}$. Adding all maps $\epsilon^J_{\infty,d}$ (44) together, one defines a map

$$
\begin{aligned}
(53) \qquad \epsilon^q_{q'} : \mathcal{C}^q_{\infty,d} &\to \mathcal{C}^{q+q'}_{\infty,\binom{q+q'}{q}d} \\
\sigma &\mapsto \sum_J \epsilon^J_{\infty,d}(\sigma).
\end{aligned}
$$

Lemma 5.1. *The following diagram commutes:*

$$
\begin{array}{ccc}
\mathcal{C}^q_{\infty,d} & \xrightarrow{\ \epsilon^q_1\ } & \mathcal{C}^{q+1}_{\infty,(q+1)d} \\
{\scriptstyle t^q_{\infty,(d,d+1)}}\downarrow & & \downarrow{\scriptstyle t^{q+1}_{\infty,(d(q+1),(d+1)(q+1))}} \\
\mathcal{C}^q_{\infty,d+1} & \xrightarrow[\ \epsilon^q_1\ ]{} & \mathcal{C}^{q+1}_{\infty,(q+1)(d+1)}.
\end{array}
$$

Definition 5.2. The maps above generate a directed system $\{\mathcal{C}^q_{\infty,d}, t^q_{\infty,(d,e)}, \epsilon^q_{q'}\}$, whose colimit we denote by $\mathcal{C}_{\mathbb{Q}}$. For this system one can also define the additive monoid $\mathcal{C}^\infty_{\infty,*_{\mathbb{Q}}} := \varinjlim_d \amalg_d \mathcal{C}^q_{\infty,d}$. Define $\#_{\mathbf{av}} : \mathcal{C}^q_{\infty,d} \times \mathcal{C}^{q'}_{\infty,d'} \to \mathcal{C}^q_{\infty,\ \binom{q+q'}{q}dd'}$ by

$$
\sigma \#_{\mathbf{av}} \sigma' = \sum_{|J|=q} sh_{J*}(\sigma \# \sigma').
$$

Proposition 5.3. 1. *The directed system above has a cofinal subsystem given by the spaces $\mathcal{C}^q_{\infty,q!}$ together with the maps $\epsilon^q_1 : \mathcal{C}^q_{\infty,q!} \to \mathcal{C}^{q+1}_{\infty,(q+1)!}$.*

2. *The maps $\#_{\mathbf{av}}$ assemble to give a map of directed systems and satisfy $\sigma\#_{\mathbf{av}}\sigma' = \sigma'\#_{\mathbf{av}}\sigma$. In particular, both colimits $\mathcal{C}_{\mathbb{Q}}$ and $\mathcal{C}^\infty_{\infty,*\mathbb{Q}}$ have the structure of commutative topological rings, whose operations are given by algebraic maps.*

Proof. Follows directly from the definitions. $\qquad\square$

Theorem 5.4. *The colimit $\mathcal{C}_{\mathbb{Q}}$ is homotopy equivalent to the rationalization of $\mathcal{C}$.*

Proof. The directed system can be visualized with the aid of the following diagram.

(54)

Note that the bottom horizontal arrow

$$\epsilon^q : \mathcal{C}^q_{\infty,\infty} = \prod_{j=1}^{q} K(\mathbb{Z},2j) \longrightarrow \mathcal{C}^{q+1}_{\infty,\infty} = \prod_{j=1}^{q+1} K(\mathbb{Z},2j)$$

is homotopic to $(q+1) \cdot \epsilon^{(q,q+1)}_{\infty,\infty}$; cf. Remark 4.4. A minor modification of the arguments in Proposition 3.1 ends the proof. $\qquad\square$
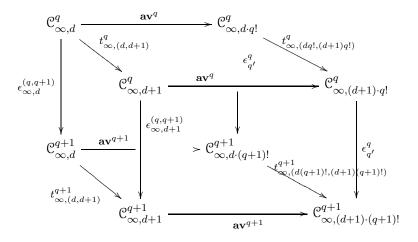
We now exhibit an explicit "algebraic" rationalization map from $\mathcal{C}$ to $\mathcal{C}_{\mathbb{Q}}$. The constructions below provide an essential link to the previous constructions with $BU(q)$ and symmetric products.

The main ingredient here is the "averaging map"

(55) $$\mathbf{av}^q : \mathcal{C}^q_{\infty,d} \to \mathcal{C}^q_{\infty,d\cdot q!},$$

which sends $\sigma \in \mathcal{C}^q_{\infty,d}$ to $\sum_{g \in \mathfrak{S}_q} g * \sigma \in \left(\mathcal{C}^q_{\infty,d \cdot q!} \right)^{\mathfrak{S}_q}$.

Lemma 5.5. *The following diagrams commute:*

and

Proof. In the first diagram, the left vertical face commutes by Lemma 4.6, and the right vertical face commutes by Lemma 5.1. The top and bottom faces commute because $\mathbf{av}^q$ can be seen as an additive endomorphism of $\mathcal{C}^q_{\infty,*}$ which sends $\mathbf{l}^\infty_q$ to $q!\mathbf{l}^\infty_q$. The commutativity of last diagram follows from an inspection of the definitions. $\square$

It follows from this lemma that the averaging maps, when put together, induce an algebraic map $\mathbf{av}^\infty : \mathcal{C} \to \mathcal{C}_{\mathbb{Q}}$, from the colimit of the left vertical faces, to the colimit of the right vertical faces. Furthermore, this map is a map of E_∞-ring spaces, for the multiplication given by the join of cycles.

Corollary 5.6. *The map $\mathbf{av}^\infty$ gives the rationalization map $\mathcal{C} \simeq \prod_{j=1}^\infty K(\mathbb{Z}, 2j) \to \mathcal{C}_{\mathbb{Q}} \simeq \prod_{j=1}^\infty K(\mathbb{Q}, 2j)$ induced by the inclusion $\mathbb{Z} \to \mathbb{Q}$.*

5.2. **Exponential maps, the case $X = \{pt\}$.** The averaging map (55) defines a morphism

(56)
$$\mathbf{av}^q : BU(q) \to \mathcal{C}^q_{\infty, q!}.$$

satisfying the following properties.

Lemma 5.7. *The averaging map (56) factors through the quotient $BU(q)/\mathfrak{S}_q$, inducing a map $\gamma^q : BU(q)/\mathfrak{S}_q \to \mathcal{C}^q_{\infty, q!}$, which makes the diagram commute*

(57)
$$
\begin{array}{ccc}
BU(q)/\mathfrak{S}_q & \xrightarrow{\gamma^q} & \mathcal{C}^q_{\infty, q!} \\
\downarrow & & \downarrow{\epsilon^q_1} \\
BU(q+1)/\mathfrak{S}_{q+1} & \xrightarrow{\gamma^{q+1}} & \mathcal{C}^{q+1}_{\infty, (q+1)!}.
\end{array}
$$

Corollary 5.8. *The maps γ^q give, by passage to colimits, commutative diagrams of E_∞-spaces*

(58)
$$
\begin{array}{ccc}
BU & \xrightarrow{c} & \mathcal{C} \\
\downarrow & & \downarrow{\mathbf{av}^\infty} \\
BU/\mathfrak{S}_\infty & \xrightarrow{\gamma^\infty} & \mathcal{C}_\mathbb{Q},
\end{array}
\quad and \quad
\begin{array}{ccc}
BU & \xrightarrow{c} & \mathcal{C}^\infty_{\infty, *} \\
\downarrow & & \downarrow{\mathbf{av}^\infty} \\
BU/\mathfrak{S}_\infty & \xrightarrow{\gamma^\infty} & \mathcal{C}^\infty_{\infty, *\mathbb{Q}},
\end{array}
$$

where $\mathbf{av}^\infty$ is described in Corollary 5.6, and c is described in Theorem 4.9(e). Hence, γ^∞ represents the rational total Chern class and is a rational homotopy equivalence.

In Proposition 2.15 we construct an algebraic map $f : \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)) \to BU/\mathfrak{S}_\infty$ with the property that $f^*(j!\mathbf{ch}j) = \mathbf{i}_{2j}$ in rational cohomology. The homotopy class of the composition

(59)
$$R \; : \; \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)) \simeq \prod_{j \geq 1} K(\mathbb{Z}, 2j) \xrightarrow{f} BU/\mathfrak{S}_\infty \xrightarrow{\gamma^\infty} \mathcal{C}_\mathbb{Q} \simeq \prod_{j \geq 1} K(\mathbb{Q}, 2j),$$

in the topological category, has the following evident interpretation as a cohomology class $\mathbf{R} \in H^*(\prod_{j \geq 1} K(\mathbb{Z}, 2j); \; \mathbb{Q})$.

Let $\Lambda = \oplus_{n=0}^\infty \Lambda_n$ denote the ring of *symmetric functions* $p(x_1, x_2, \dots)$ on infinitely many variables, where Λ_n denotes the functions of degree n. Here we follow the notation of [Ful97]. Let $e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$ be the k-th *elementary symmetric function* and $p_k = \sum_i x_i^k$ be the k-th *Newton power sum*. It is well-known that Λ is a polynomial ring over $\mathbb{Z}$ in the variables $\{e_1, e_2, \dots\}$ and that $\Lambda \otimes \mathbb{Q}$ is a polynomial ring over $\mathbb{Q}$ in the variables $\{p_1, p_2, \dots\}$. In particular, there are universal polynomials $R_j(Y_1, \dots, Y_j)$ with rational coefficients such
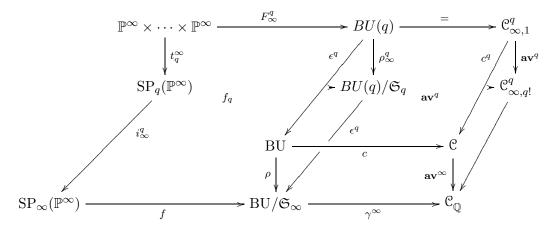
that

(60)
$$e_k = R_j(p_1, \ldots, p_j);$$

cf. [Ful97].

Proposition 5.9. *Let* $\mathbf{R}_j \in H^{2j}(\prod_{j \geq 1} K(\mathbb{Z}, 2j), \mathbb{Q})$ *be the* j-*th component of* $\mathbf{R}$. *Then* $\mathbf{R}_j = R_j(\mathbf{i}_2, \ldots, \mathbf{i}_{2j})$, *where* R_j *is the universal polynomial in (60) and* $\mathbf{i}_{2k}$ *is the rational fundamental class (20).*

Proof. Consider the following diagram.

It follows from definitions that $(F_\infty^q)^* \circ (\epsilon^q)^* \circ \rho^*(\mathbf{c}_j) = \sum_{1 \leq i_1 < \cdots i_j \leq q} x_{i_1} \cdots x_{i_j} = e_j(x_{i_1}, \ldots, x_{i_j})$ and that $(F_\infty^q)^* \circ (\epsilon^q)^* \circ \rho^*(j!\mathbf{ch}j) = x_1^j + \cdots + x_q^j$. Hence, one has $\mathbf{c}_j = R_j(1!\mathbf{ch}1, \ldots, j!\mathbf{ch}j)$.

Therefore,

$$\mathbf{R}_j = f^*(\gamma^\infty)^*(\mathbf{i}_{2j}) = f^*(\mathbf{c}_j)$$
$$= f^*(R_j(1!\mathbf{ch}1, \ldots, j!\mathbf{ch}j)) = R_j(f^*(1!\mathbf{ch}1), \ldots, f^*(j!\mathbf{ch}j))$$
$$= R_j(\mathbf{i}_2, \cdots, \mathbf{i}_{2j});$$

where the last equality comes from Theorem 3.7.   □

5.3. **Exponential maps, the case X arbitrary.** Throughout the rest of this paper, X is a smooth projective algebraic variety. The complex suspension $\Sigma^X : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)))^+ \to \mathfrak{Mor}(X, \mathcal{C})^+$ (cf. Remark 4.10) provides a filtration preserving additive monoid morphism. In particular, it gives an equivalence

(61)
$$\Sigma_\mathbb{Q}^X : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q})^+ \to \mathfrak{Mor}(X, \mathcal{C})_\mathbb{Q}^+$$

compatible with the various geometric constructions described in previous sections.

Using the map γ^∞, cf. (58), one obtains a composition of maps $\mathfrak{Mor}(X, \mathrm{BU}/\mathfrak{S}_\infty) \to \mathfrak{Mor}(X, \mathfrak{C}^\infty_{\infty,*_\mathbb{Q}}) \to \mathfrak{Mor}(X, \mathfrak{C}^\infty_{\infty,*_\mathbb{Q}})^+$ which factors through

$$\mathfrak{Mor}(X, \mathrm{BU}/\mathfrak{S}_\infty) \to \mathfrak{Mor}(X, \mathfrak{C}^\infty_{\infty,*_\mathbb{Q}})_1^+ \to \mathfrak{Mor}(X, \mathfrak{C}^\infty_{\infty,*_\mathbb{Q}})^+,$$

where $\mathfrak{Mor}(X, \mathfrak{C}^\infty_{\infty,*_\mathbb{Q}})_1^+ := \Phi^{-1}(1)$ for the natural augmentation $\Phi : \mathfrak{Mor}(X, \mathfrak{C}^\infty_{\infty,*_\mathbb{Q}})^+ \to \mathbb{Z}$.

One can use the complex suspension $\Sigma^X_\mathbb{Q}$ equivalence, along with the arguments preceding Definition 4.12 and Proposition 3.1 applied to the filtration $\left(\mathrm{II}_d\ \mathfrak{C}^\infty_{n,d}\right) \subset \left(\mathrm{II}_d\ \mathfrak{C}^\infty_{n+1,d}\right) \subset \cdots \subset \mathfrak{C}^\infty_{\infty,*}$, to obtain a map $\gamma^\infty_{X,\mathbb{Q}}$ which fits into a homotopy commutative diagram

(62)

$$
\begin{array}{ccc}
\mathfrak{Mor}(X, BU)^+ & \xrightarrow{\ c_X\ } & \mathfrak{Mor}(X, \mathfrak{C}^\infty_{\infty,*})_1^+ \\
{\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \mathbf{av}^\infty_X} \\
\mathfrak{Mor}(X, \mathrm{BU}/\mathfrak{S}_\infty)^+ & \xrightarrow{\ \gamma^\infty_X\ } & \mathfrak{Mor}(X, \mathfrak{C}^\infty_{\infty,*_\mathbb{Q}})_1^+ \\
\downarrow & \nearrow_{\gamma^\infty_{X,\mathbb{Q}}} & \\
\mathfrak{Mor}(X, \{\mathrm{BU}/\mathfrak{S}_\infty\}_\mathbb{Q})^+ & &
\end{array}
$$

It follows from Remark 50 that $\mathfrak{Mor}(X, \mathfrak{C}^\infty_{\infty,*_\mathbb{Q}})_1^+$ can be identified with the rationalization of $\mathfrak{Mor}(X, \mathfrak{C})^+$, and hence the composition $\gamma^\infty_X \circ \rho$ gives the rational total Chern class map.

Our goal now is to provide an alternative description of the composition

$$\mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)_\mathbb{Q})^+ \xrightarrow{f^X_\mathbb{Q}} \mathfrak{Mor}(X, \{\mathrm{BU}/\mathfrak{S}_\infty\}_\mathbb{Q})^+ \xrightarrow{\gamma^\infty_{X,\mathbb{Q}}} \mathfrak{Mor}(X, \mathfrak{C}_\mathbb{Q})^+.$$

First, let $\pi^X_j : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q}) \to \mathfrak{Mor}(X, \mathrm{SP}_\infty(S^{2j})_\mathbb{Q})$ be the composition of the splitting map (30) with the projection onto the j-th factor. Then define

$$i^X_j : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q})^+ \to \mathfrak{Mor}(X, \mathfrak{C}_\mathbb{Q})^+$$

as the composition

$$\mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)_\mathbb{Q})^+ \xrightarrow{i^X_j} \mathfrak{Mor}(X, \mathrm{SP}_\infty(S^{2j})_\mathbb{Q}) \hookrightarrow \prod_p \mathfrak{Mor}(X, \mathrm{SP}_\infty(S^{2p})_\mathbb{Q})$$

$$\xrightarrow{(sp^X_\mathbb{Q})^{-1}} \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty)_\mathbb{Q})^+ \xrightarrow{\Sigma^X_\mathbb{Q}} \mathfrak{Mor}(X, \mathfrak{C}_\mathbb{Q})^+.$$

Then, let $R_p(Y_1, \ldots, Y_p)$ be the polynomial defined in 60. Since $\mathfrak{Mor}(X, \mathfrak{C}_\mathbb{Q})^+$ is an abelian topological ring one can use R_p and the maps i^X_j introduced above to define

(63) $$R_p(i^X_1, \ldots, i^X_p) : \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q})^+ \to \mathfrak{Mor}(X, \mathfrak{C}_\mathbb{Q})^+.$$

These maps then assemble to give unique homotopy class of maps

$$R^X \ : \ \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q})^+ \to \mathfrak{Mor}(X, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q})^+$$

satisfying $\pi_p^X \circ R^X = \pi_p^X \circ \left\{ \Sigma_\mathbb{Q}^X \right\}^{-1} \circ R_p(i_1^X, \dots, i_p^X)$. This construction is functorial on X.

Theorem 5.10. *The composition $\left\{ \Sigma_\mathbb{Q}^X \right\}^{-1} \circ \gamma_\mathbb{Q}^X \circ f_\mathbb{Q}^X$ is homotopic to R^X.*

Proof. This follows from the universal case $X = \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q}$ and the fact that the forgetful functor from $\mathfrak{Mor}(\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q}, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q})$ to $\mathfrak{Map}(\mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q}, \mathrm{SP}_\infty(\mathbb{P}(\mathbb{C}^\infty))_\mathbb{Q})$ is a homotopy equivalence. □

Corollary 5.11. *The p-th component $c_X^{p,j}$ of the j-th Chern class*

$$c_X^j \ : \ \tilde{\mathcal{K}}_{hol}^{-j}(X) \to \prod_{p \geq 1} L^p H^{2p-j}(X)$$

is given by the universal polynomial

$$c_X^{p,j} = \frac{1}{p!} R_p(1! ch_{1,p}^X, \dots, p! ch_{j,p}^X).$$

on the Chern characters.

Proof. This is similar to the case $X = \{pt\}$. The theorem above gives an equivalence

$$\left(\Sigma_\mathbb{Q}^X \right)^{-1} \circ \gamma_{X,\mathbb{Q}}^\infty \sim R^X \circ \left(f_\mathbb{Q}^X \right)^{-1} = \left(R^X \circ \mathbf{e}^{-1} \right) \circ \left(\mathbf{e} \circ \left(f_\mathbb{Q}^X \right)^{-1} \right).$$

Composing both sides with the projections $\mathfrak{Mor}(X, BU)^+ \xrightarrow{\rho^X} \mathfrak{Mor}(X, BU/\mathfrak{S}_\infty)^+ \xrightarrow{r_B^X} \mathfrak{Mor}(X, \{BU/\mathfrak{S}_\infty\}_\mathbb{Q})^+$, and using the definitions of the rational Chern classes and the Chern character, concludes the proof. □

APPENDIX A. GROUP COMPLETIONS OF MORPHISM SPACES

Throughout this appendix, B and S will denote abelian topological monoids, with operations $\mu_B : B \times B \to B$ and $\mu_S : S \times S \to S$, respectively, which satisfy the conditions of Remark 2.2. In other words, both B and S are colimits of algebraic varieties whose operations are induced by algebraic maps of the corresponding directed systems. Given any algebraic variety X, or a colimit of varieties, the morphism spaces $\mathfrak{Mor}(X, B)$ and $\mathfrak{Mor}(X, S)$ become abelian topological monoids under pointwise addition; cf. Remark 2.2.

Consider an arbitrary topological monoid M. If (A, M, B) is a triple where A is a right M-space and B is a left M-space, then one can construct the *triple bar construction* $\mathcal{B}(A, M, B)$. This is a functorial construction on such triples satisfying the following properties.

1. If (A, M, B) is such a triple and M acts trivially on C, then $\mathcal{B}(C \times A, M, B) = C \times \mathcal{B}(A, M, B)$, where M acts diagonally on $C \times A$; cf. [May75].

2. $\mathcal{B}(*, M, *) = BM$ is the classifying space of M and the map $EM := \mathcal{B}(M, M, *) \to BM$ induced by the obvious map of triples $(M, M, *) \to (*, M, *)$ is the universal quasifibration for M; cf. [May75].

3. If M is abelian, then BM is an abelian monoid and so is ΩBM under pointwise addition.

4. If M is abelian, and $(M \times M, M, *)$ is the triple where M acts diagonally on $M \times M$, then ΩBM is naturally homotopy equivalent to $\mathcal{B}(M \times M, M, *)$, which is the homotopy quotient of $M \times M$ by the diagonal action; cf. [LF93,]. Furthermore, the involution $M \times M \to M \times M$ sending (m, n) to (n, m) induces an involution $\iota_M : \mathcal{B}(M \times M, M, *) \to \mathcal{B}(M \times M, M, *)$ which is natural on M and corresponds to giving the "inverse" of an element. In other words, $id + \iota_M$ is naturally homotopic to zero.

The last property allows one to use the model $M^+ = \mathcal{B}(M \times M, M, *)$ for the homotopy theoretic group-completion ΩBM of an abelian monoid M.

Let B and S be monoids as above, and let X be an algebraic variety. The composition map

(64) $$\psi : \ \mathcal{M}\mathfrak{or}(X, B) \times \ \mathcal{M}\mathfrak{or}(B, S) \longmapsto \ \mathcal{M}\mathfrak{or}(X, S)$$

$$(f, a) \longmapsto a \circ f$$

induces an evident map of triples

$$\psi : (\ \mathcal{M}\mathfrak{or}(X, B) \times (\ \mathcal{M}\mathfrak{or}(B, S) \times \ \mathcal{M}\mathfrak{or}(B, S)), \ \mathcal{M}\mathfrak{or}(B, S), *) \to (\ \mathcal{M}\mathfrak{or}(X, S) \times \ \mathcal{M}\mathfrak{or}(X, S), \ \mathcal{M}\mathfrak{or}(X, S), *),$$

where the monoid $\ \mathcal{M}\mathfrak{or}(B, S)$ acts on $\ \mathcal{M}\mathfrak{or}(X, B)$ trivially, and the other monoid actions are diagonal actions. Therefore, ψ induces a map of respective triple bar constructions

(65) $$\Psi : \ \mathcal{M}\mathfrak{or}(X, B) \times \ \mathcal{M}\mathfrak{or}(B, S)^+ \to \ \mathcal{M}\mathfrak{or}(X, S)^+,$$

which makes the following diagram commute,

$$
\begin{array}{ccc}
\mathcal{M}\mathfrak{or}(X, B) \times \ \mathcal{M}\mathfrak{or}(B, S) & \xrightarrow{\ \psi\ } & \mathcal{M}\mathfrak{or}(X, S) \\
{\scriptstyle id \times u_1} \big\downarrow & & \big\downarrow {\scriptstyle u_2} \\
\mathcal{M}\mathfrak{or}(X, B) \times \ \mathcal{M}\mathfrak{or}(B, S)^+ & \xrightarrow{\ \Psi\ } & \mathcal{M}\mathfrak{or}(X, S)^+,
\end{array}
$$

where u_1 and u_2 denote the universal maps from the monoids to their group-completion.

Definition A.1. Let B, S and X be as above. Given $\alpha \in \mathfrak{Mor}(B,S)^+$ define

$$\alpha_*^X \; : \; \mathfrak{Mor}(X,B) \to \mathfrak{Mor}(X,S)^+$$

by $\alpha_*^X(f) = \Psi(f,\alpha)$, where Ψ is defined in (65).

If $\tau \in \mathfrak{Mor}(M,N)$ is a monoid morphism, then it naturally induces a monoid morphism

$$\tau_+^X : \mathfrak{Mor}(X,M)^+ \to \mathfrak{Mor}(X,N)^+,$$

due to the functoriality of bar constructions. Examples of interest are given by multiplication morphisms $\mu_B : B \times B \to B$ and $\mu_S : S \times S \to S$, when B and S are abelian monoids as above.

Proposition A.2.

   a: *The assignment $X \mapsto \alpha_*^X$ is contravariantly functorial on X.*
   b: *Given $\alpha, \beta \in \mathfrak{Mor}(B,S)^+$, if $[\alpha] = [\beta] \in \pi_0\, \mathfrak{Mor}(B,S)^+$, then α_*^X is naturally homotopic to β_*^X.*

Proof. The first assertion is evident.

   If γ_t is a path between α and β, then $\Psi(-,\gamma_t)$ provides the natural homotopy between α_*^X and β_*^X. $\qquad\square$

Remark A.3. For an explicit and natural description of α_*^X, up to a homotopy natural on X, one proceeds as follows. Write $[\alpha] \in \pi_0(\,\mathfrak{Mor}(B,S)^+) = (\pi_0(\,\mathfrak{Mor}(B,S)))^+$ as a difference of two homotopy classes $[a^+] - [a^-]$, where $a^+, a^- \in \mathfrak{Mor}(S,B)$. Then, for each X one has a map of triples

$$(\,\mathfrak{Mor}(X,B), *, *) \longrightarrow (\,\mathfrak{Mor}(X,S) \times \mathfrak{Mor}(X,S),\, \mathfrak{Mor}(X,S), *)$$
$$(f,*,*) \longmapsto (a^+ \circ f \; \times \; a^- \circ f,\; 0,\; *),$$

inducing a map $\mathfrak{Mor}(X,B) \to \mathfrak{Mor}(X,S)^+$ which is in the same homotopy class of α_*^X. This is equivalent to say that α_*^X is naturally homotopic to $a_*^{+,X} + \iota \circ a_*^{-,X}$, where ι is the involution of $\mathfrak{Mor}(X,S)^+$ described in the properties of the bar construction above.

   The natural map $\mathfrak{Mor}(B,S) \times \mathfrak{Mor}(B,S) \to \mathfrak{Mor}(B \times B, S \times S)$ is a monoid morphism and hence it induces a monoid morphism

(66) $$\mathfrak{Mor}(B,S)^+ \times \mathfrak{Mor}(B,S)^+ \to \mathfrak{Mor}(B \times B, S \times S)^+$$

between their completions. In particular, given $\alpha \in \operatorname{Mor}(B,S)^+$, we denote by $\alpha \times \alpha$ both the obvious element in $\operatorname{Mor}(B,S)^+ \times \operatorname{Mor}(B,S)^+$ and its image in $\operatorname{Mor}(B \times B, S \times S)^+$ under the map above.

Definition A.4. We say that an element $\alpha \in \operatorname{Mor}(B,S)^+$ is a homotopy homomorphism if $\mu_{S,+}^{B \times B}(\alpha \times \alpha)$ and $\alpha_*^{B \times B}(\mu_B)$ represent the same class in $\pi_0(\operatorname{Mor}(B \times B, S)^+)$.

This definition is the translation to the completed level of the statement that an element $\tau \in \operatorname{Mor}(B,S)$ is an H-space map. The following result follows directly from the definitions.

Proposition A.5. *Let* $f_1 : \operatorname{Mor}(B,S)^+ \to \operatorname{Map}(B,S)^+$ *and* $f_2 : \operatorname{Mor}(B \times B, S)^+ \to \operatorname{Map}(B \times B, S)^+$ *denote the forgetful functors, and let* $\alpha \in \operatorname{Mor}(B,S)^+$ *be such that* $f_1(\alpha)$ *is a homotopy homomorphism in the topological category. If* f_1 *and* f_2 *are are homotopy equivalences, then* α *is a homotopy homomorphism; cf. Definition A.4.*

Proposition A.6. *Let* X *be an algebraic variety and* $\alpha \in \operatorname{Mor}(B,S)^+$, *with* B *and* S *abelian monoids as above. If* α *is a homotopy homomorphism, then* $\alpha_*^X : \operatorname{Mor}(X,B) \to \operatorname{Mor}(X,S)^+$ *is an* H-*space map. Furthermore, the homotopy making the diagram below commute is functorial on* X.

$$
\begin{array}{ccc}
\operatorname{Mor}(X,B) \times \operatorname{Mor}(X,B) & \xrightarrow{\ \mu_B^X\ } & \operatorname{Mor}(X,B) \\
{\scriptstyle \alpha_*^X \times \alpha_*^X} \downarrow & & \downarrow {\scriptstyle \alpha_*^X} \\
\operatorname{Mor}(X,S)^+ \times \operatorname{Mor}(X,S)^+ & \xrightarrow{\ \mu_{B,+}^X\ } & \operatorname{Mor}(X,S)^+
\end{array}
$$

Proof. It follows from the definitions that, given $f, g \in \operatorname{Mor}(X,B)^+$, one has

$$
\alpha_*^X \circ \mu_B^X(f \times g) = \left\{ \alpha_*^{B \times B}(\mu_B) \right\}_*^X (f \times g)
$$

and

$$
\mu_{S,+}^X \circ (\alpha_*^X \times \alpha_*^X)(f \times g) = \left\{ \mu_{S,+}^{B \times B}(\alpha \times \alpha) \right\}_*^X (f \times g).
$$

By hypothesis one has that $\alpha_*^{B \times B}(\mu_B)$ is homotopic to $\mu_{S,+}^{B \times B}(\alpha \times \alpha)$, and the result now follows from Proposition A.2. $\qquad\square$

Remark A.7. We must point out that all constructions of this Appendix could have been made replacing $\operatorname{Mor}(,)$ by $\operatorname{Map}(,)$, and that the forgetful functor $\operatorname{Mor}(,) \to \operatorname{Map}(,)$ would then induce natural transformations between all functors studied in the Appendix.

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